CONVEX DUALITY WITH TRANSACTION COSTS

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Abstract. Convex duality for two two different super–replication problems in a continuous time financial market with proportional transaction cost is proved. In this market, static hedging in a finite number of options, in addition to usual dynamic hedging with the underlying stock, are allowed. The first one of the problems considered is the model–independent hedging that requires the super–replication to hold for every continuous path. In the second one the market model is given through a probability measure $\mathbb{P}$ and the inequalities are understood $\mathbb{P}$ almost surely. The main result, using the convex duality, proves that the two super–replication problems have the same value provided that $\mathbb{P}$ satisfies the conditional full support property. Hence, the transaction costs prevents one from using the structure of a specific model to reduce the super–replication cost.

1. Introduction

The problem of super-replication is a convex optimization problem in which the investor minimizes the cost of a portfolio among those satisfying the hedging constraints. In the classical case the financial market is frictionless and the investors can buy or sell any quantity of the stocks and other financial instruments at the same price. Then, the corresponding problem is linear and the optimization problem is in fact an infinite dimensional linear program. In the quantitative finance literature, this problem is well studied and is known to be related to arbitrage. One central result is a convex duality result, which contains deep financial insights including the fundamental theorem of asset pricing.

In the celebrated papers [9, 10, 18] the financial market is modelled through a probability measure $\mathbb{P}$ that describes the future movements of the stock prices in the time interval $[0, T]$. The stock price process S and the measure $\mathbb{P}$ are defined on a probability space Ω and a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{\{t\in[0,T]\}}$. The main object of study is an uncertain liability that will be revealed in the future. It is usually modelled through a $\mathcal{F}_T$ measurable random variable ξ and the main goal is to reduce the risk related to ξ by appropriately trading in the financial market. The investment opportunities is given abstractly through a linear set $\hat{\mathcal{A}}$ denoting the set of all admissible portfolios π with a final portfolio value Z^π_T at time T. Then, the

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super-replication problem is to minimize the cost among all portfolios that reduces the risk related to the liability ξ to zero. Mathematically,

$$(1.1) \qquad V(\xi) := \inf \left\{ \mathcal{L}(\pi) \ : \ \exists \pi \in \hat{\mathcal{A}} \text{ such that } Z_T^{\pi} \geq \xi, \ \ \mathbb{P} - a.s. \right\},$$

where $\mathcal{L}(\pi) \in \mathbb{R}$ is the cost of the portfolio π. Once a market model is fixed through a probability measure $\mathbb{P}$, then all statements are supposed to be understood $\mathbb{P}$-almost surely. Hence, the only role of the probability measure $\mathbb{P}$ is to describe the null sets or equivalently all impossible future scenarios. Any other probability measure that is equivalent to $\mathbb{P}$ (i.e., any measure with the same null sets) would yield the same super-replication cost. This problem is studied extensively when the market is frictionless or equivalently $\mathcal{L}$ is linear and when only the adapted dynamic trading of the stock without constraints is considered. Under no-arbitrage type assumptions and mild technical integrability conditions, the convex dual is the following maximization problem,

$$D(\xi) := \sup_{\mathbb{Q} \in \mathcal{Q}} \ \mathbb{E}_{\mathbb{Q}} \left[\xi \right],$$

where $\mathcal{Q}$ is the set of all "martingale" measures that are equivalent to $\mathbb{P}$. Precise statements in continuous time are technical and we refer the reader to the seminal paper of Delbaen & Schachermayer [10].

These classical results were then extended to markets with trading frictions. It is shown that super-replication in markets with (proportional) transaction costs is prohibitively costly as first proved in Soner Shreve & Cvitanic [23] and later generalized in Leventhal & Skorohod [19], Cvitanic, Pham & Touzi [8], Bouchard & Touzi [6], Jakubenas, Levental, & Ryznar [17], Guasoni, Rasonyi & Schachermayer [15], Blum [4] and for the game options in Dolinsky [11]. In all of these examples, the super-replication cost is minimized among all "trivial" strategies. Hence, the investor does not benefit from dynamic hedging when the objective is to super replicate with certainty. Also in all of these examples not the null sets of $\mathbb{P}$ but rather the support of it is important. The related question of fundamental theorem of asset pricing and super-heding duality with a given $\mathbb{P}$ is studied by Schachermayer [20, 21] and the references therein.

One may reduce the hedging cost by including liquid derivatives in the super-replicating portfolio. In particular, this might be the case for semi-static hedging which is detailed in the next section. Namely, the investor is allowed to take static positions in a finite number of options (written on the underlying asset) with initially known prices. In addition to these static option positions, the stock is also traded dynamically and all of these trades are subject to proportional transaction costs. In terms of the above notation, the set $\hat{\mathcal{A}}$ of admissible portfolios is enlarged by static option trades but the transaction costs make the cost functional $\mathcal{L}$ to be convex rather than to be linear as in the classical papers. We refer the reader to the survey of Hobson [16], a recent paper of the authors [14] and the references therein for information on semi-static hedging in continuous time.

While the model-independent approach with semi-static hedging received considerable attention in recent years, there are only few results for such markets with friction. Indeed, recently the authors proved a model independent duality result for semi–static hedging with transaction costs in discrete time [13]. Again in discrete time a fundamental theorem asset pricing was studied in Bayraktar & Zhang [2] and in Bouchard & Nutz [5] in markets with transaction costs. These later papers

consider the quasi-sure criterion given by a set of probabilistic models. To the best of our knowledge, in continuous time semi–static hedging with transaction costs under model uncertainty has not yet been studied.

In this paper, we consider a continuous time financial market which consists of one risky asset with continuous paths. In such a financial market we study two super–replication problems of a given (path dependent) European option. We assume that the dynamic hedging of the stock as well as the static option trading are subject to transaction fees. In the first problem, the market model is given through a probability measure $\mathbb{P}$. Then, the optimization problem corresponds to a straightforward extension of (1.1). The second one is the model–independent problem referring to super–replication for all continuous stock price processes. Namely, in (1.1) we require the inequality $Z^{\pi}_T \geq \xi$ to hold not $\mathbb{P}$-almost surely but rather *for every possible* stock price path. These definitions are given in the subsection 2.5 below.

Our main result Theorem 2.7 states that these two problems described above have the same value provided that the distribution $\mathbb{P}$ of the stock price process satisfies the conditional full support property, see Definition 2.6 below. Hence, in the presence of transaction costs the knowledge of the model does not reduce the super–replication cost. This explains the earlier results on super-replication with friction and why the optimal hedge in these examples are the trivial ones.

Theorem 2.7 is proved under regularity Assumptions 2.1, 2.2 and a no-arbitrage type of condition Assumption 2.3, below. However, we do not assume any admissibility conditions on the portfolio. Furthermore, we provide a duality result for the mutual value in terms of consistent price systems on the space of continuous functions that are consistent with the option prices. This duality is very similar to the one proved in discrete time in [13].

The proof of Theorem 2.7 is completed in four major steps. First, we reduce the problem to bounded payoffs by applying the pathwise inequalities which were obtained in Acciaio *et.al.* [1] and earlier by Burkholder [7]. In the second step, we obtain a lower bound for the super–replication cost in the case where the model is given. This bound is expressed in terms of modified model–free super–replication problems with appropriately lowered rate of transaction costs. The third step is to derive an upper bound for the model–free problem. This step is done by applying the recent results of Schachermayer [21] together with a lifting procedure similar to the one developed in our earlier work [12]. The last step is a probabilistic proof for the equality between (the asymptotic behaviour of) the lower and the upper bounds.

The paper is organised as follows. Main results are formulated in the next section. In Section 3, we reduce the problem to bounded claims. A lower bound for the super–replication price in a given model is obtained in Section 4. Section 5 derives an upper bound for the model–free super–replication price. The last section is devoted to the proof of the equality between the lower and the upper bounds.

2. Preliminaries and main results

2.1. **Market and Notation.** The financial market consists of a savings account which is normalized to unity $B_t \equiv 1$ by discounting and of a risky asset S_t, $t \in [0,T]$, where $T < \infty$ is the maturity date. Let $s := S_0 > 0$ be the initial stock price and without loss of generality set $s = 1$. We assume that the risky asset could be any continuous process with this initial data.

In the sequel, we use the following notations. For $s \geq 0$, $t \in [0,T)$, we set

$$\mathcal{C}^+_s[t,T] := \{ \ f : [t,T] \to [0,\infty) \mid f \text{ is continuous}, \ f(t) = s \ \},$$
$$\mathcal{C}^+[t,T] := \bigcup_{s \geq 0} \mathcal{C}^+_s[t,T]$$

and for $s > 0$,

$$\mathcal{C}^{++}_s[t,T] := \left\{ \ f \in \mathcal{C}^+_s[t,T] \mid f(u) > 0, \ \forall u \in [t,T] \ \right\},$$
$$\mathcal{C}^{++}[t,T] := \bigcup_{s > 0} \mathcal{C}^{++}_s[t,T].$$

Then,

$$\Omega := \mathcal{C}^{++}_1[0,T]$$

represents the set of all possible stock prices or the probability space. We let $\mathbb{S} = (\mathbb{S}_t)_{0 \leq t \leq T}$ be the canonical process given by $\mathbb{S}_t(\omega) := \omega_t$, for all $\omega \in \Omega$ and $\mathbb{F}_t := \sigma(\mathbb{S}_s, \ 0 \leq s \leq t)$ be the canonical filtration (which is not right continuous). We say that a probability measure $\mathbb{Q}$ on the space $(\Omega, \mathbb{F})$ is a martingale measure, if the canonical process $(\mathbb{S}_t)_{t=0}^T$ is a martingale with respect to $\mathbb{Q}$.

Further we let

$$\mathbb{D}[0,T] := \{ \ f : [t,T] \to [0,\infty) \mid f \text{ is } c\grave{a}dl\grave{a}g \ \},$$

be the Skorokhod space of *càdlàg* functions with the usual sup-norm

$$\|v\| := \sup_{0 \leq t \leq T} |v_t|.$$

2.2. **The claim and its regularity.** We model the liability of the claim through a deterministic map of the whole stock price process. Indeed, for a given deterministic map

$$G : \mathbb{D}[0,T] \to \mathbb{R}_+,$$

a general path dependent European option has the payoff $\xi = G(S)$. Hence, although we consider only continuous stock price processes, we implicitly assume that the option is defined for all bounded measurable maps.

Our regularity assumption on the payoff is the same as the one used in [12]. For the convenience of the reader we briefly review this assumption, but refer to [12] for an extended discussion and its connection with the Skorokhod metric. In particular, all options on the running maximum and Asian type options satisfy it. We make the following standing assumption on G.

Assumption 2.1. *We assume that there exists a constant $L > 0$ satisfying,*
i.

$$|G(\omega) - G(\tilde{\omega})| \leq L\|\omega - \tilde{\omega}\|, \quad \omega, \tilde{\omega} \in \mathcal{D}[0,T],$$

ii. and

$$|G(\upsilon) - G(\tilde{\upsilon})| \leq L\|\upsilon\| \sum_{k=1}^{n} |\Delta t_k - \Delta \tilde{t}_k|,$$

for every piecewise constant functions $\upsilon, \tilde{\upsilon} \in \mathcal{D}[0,T]$ *of the form*

$$\upsilon_t = \sum_{i=0}^{n-1} \upsilon_i \chi_{[t_i, t_{i+1})}(t) + \upsilon_n \chi_{[t_n, T]}(t) \text{ and } \tilde{\upsilon}_t = \sum_{i=0}^{n-1} \upsilon_i \chi_{[\tilde{t}_i, \tilde{t}_{i+1})}(t) + \upsilon_n \chi_{[\tilde{t}_n, T]}(t),$$

where $t_0 = 0 < t_1 < ... < t_n < T$, $\tilde{t}_0 = 0 < \tilde{t}_1 < ... < \tilde{t}_n < T$ *are two partitions and as usual* $\Delta t_k := t_k - t_{k-1}$, $\Delta \tilde{t}_k := \tilde{t}_k - \tilde{t}_{k-1}$, χ_A *is the characteristic function.*

2.3. **Static Positions.** Next we describe the assumptions on the static options. We assume that there are N many options

$$f_1, ..., f_N : \mathbb{D}[0,T] \to \mathbb{R}$$

that are initially available for static hedging. These options may be path dependent. We assume that their prices $\mathcal{L}_1, ..., \mathcal{L}_N \in \mathbb{R}$ are known and that we can take *static long positions* on these options. In this context, short positions can also be allowed by including the negative of the options, but the prices of these two (option and its negative) should add up to a positive value equaling the bid-ask spread on this option. Set

$$\mathcal{F}(S) := (1, f_1(S), ..., f_N(S)) \quad \text{and} \quad \mathcal{L} := (1, \mathcal{L}_1, ..., \mathcal{L}_N),$$

where the first function which is identically equal to one stands for investment in the non risky asset and we assume that the investor can take long or short positions only in this option. But as discussed before, we allow only long positions in the other options. Thus, a static position in the these options is represented by $c \in \mathbb{R} \times \mathbb{R}_+^N$ indicating an investment of a European option with the payoff $c \cdot \mathcal{F}(S)$ for the price

$$\mathcal{L}(c) := c \cdot \mathcal{L}, \tag{2.1}$$

where $\cdot$ denotes the standard inner product of $\mathbb{R}^{N+1}$.

We assume that the static options satisfy some regularity assumptions and one of the static options has a super quadratic growth. More precisely, we assume the following.

Assumption 2.2. *Functions* $f_1, ..., f_{N-1}$ *satisfy Assumption 2.1. We also assume that if* f_i *is path dependent (i.e. do not depend only on the value of the stock at the maturity) then it is bounded. For* $i = N$*, we assume that* $f_N(\omega) = q(\omega_T)$ *where* $q : \mathbb{R}_+ \to \mathbb{R}_+$ *is a convex function satisfying*

$$|q(x) - q(y)| \leq L|x-y| \left(1 + \frac{q(x)}{x} + \frac{q(y)}{y}\right), \quad \forall x, y > 0$$

and

$$\liminf_{x \to \infty} \frac{q(x)}{x^2} > 0. \tag{2.2}$$

□

Since we consider hedging under proportional transaction costs, it is reasonable to assume that the options $f_1(S), ..., f_N(S)$ are also subject to transaction costs. This together with no-arbitrage considerations (see also [2, 5]) leads us to the following assumption.

Assumption 2.3. *There exists a martingale measure* $\mathbb{Q}$ *on the canonical space* $(\Omega, \mathbb{F})$ *such that*

$$\mathbb{E}_{\mathbb{Q}}\left[f_i(\mathbb{S})\right] < \mathcal{L}_i, \quad \forall i = 1, \ldots N,$$

where $\mathbb{E}_{\mathbb{Q}}$ *denotes the expectation with respect to the probability measure* $\mathbb{Q}$*.*

□

Remark 2.4 (**Comments on the assumptions).** In this paper we assume that there are only finitely many static options. This setup is different from the one in [12, 13, 14] where we assumed that the set of static options equals to $\{f(S_T) : f : \mathbb{R}_+ \to \mathbb{R}\}$ (and includes power options). The present assumptions seem to be more realistic. We still assume that we have an option with super quadratic payoff f_N. This is needed for reducing the problem to bounded claims and for dealing with the hedging and the pricing error estimates arising in our discretization procedure. In fact, it is sufficient to include an option with super linear payoff, however for the simplicity of computations we assume super–quadratic growth. Since the main focus of this paper is the equivalence between two different super-replication problems, we do not seek the most general assumptions on the static options. It is plausible that the main result holds under weaker assumptions. In particular, for bounded claims one might be able to avoid the use of the quadratic option as in [13].

The second assumption states that there exist a linear pricing rule that is consistent with the observed option data. This implies in particular no-arbitrage in this market. Also the strict inequality implies that the options are subject to proportional transaction costs. The equivalence of no-arbitrage and the existence of such measures is in fact a difficult question. Only recently several discrete time results in this direction were proved in [2, 5].

□

2.4. **Hedging with transaction costs.** We continue by describing the continuous time trading with proportional transaction costs, in the underlying asset S. Let $\kappa \in (0, 1)$ be the proportional transaction cost rate. Denote by γ_t the number of shares of the risky asset in the portfolio π at moment of time t before the transaction at this time. Due to transaction costs it has to be of bounded variation. Hence, we assume that the process $\gamma = \{\gamma_t\}_{t=0}^{T}$ is an adapted process (to the raw filtration generated by the stock price process) of bounded variation with left continuous paths with $\gamma_0 = 0$. Let

$$\gamma_t = \gamma_t^+ - \gamma_t^-$$

be a decomposition of γ into positive and negative variations. Namely, γ_t^+ denotes the cumulative number of stocks purchased up to time t not including the transfers made at time t and respectively, γ_t^-, denotes the cumulative number of stocks, sold up to time t again not including the transfers made at time t. Let $\mathcal{A}$ be the set of all such processes.

In this financial market, a *hedge* is a pair $\pi = (c, \gamma) \in \hat{\mathcal{A}} := \mathbb{R} \times \mathbb{R}_+^N \times \mathcal{A}$ and the corresponding *portfolio liquidation value* at the maturity date T is given by

$$\begin{aligned} Z_T^\pi(S) \quad := \quad & c \cdot \mathcal{F}(S) + \left[\gamma_T - \kappa|\gamma_T|\right] \; S_T \\ & +(1-\kappa)\int_{[0,T]} S_u \; d\gamma_u^- - (1+\kappa)\int_{[0,T]} S_u \; d\gamma_u^+, \end{aligned}$$

where the above integrals are the standard Stieltjes integrals and $\mathcal{F}(S)$ is as in subsection 2.3. Notice that the term $-\kappa|\gamma_T|\, S_T$ in the first line is due to liquidation cost at maturity. The *cost of this portfolio* $\pi = (c, \gamma)$ is equal to $\mathcal{L}(c)$ as defined in (2.1).

2.5. **Super–replication problems.** In this subsection, we introduce two super–replication problems. For the liability $\xi = G(S)$, the model–free super–replication cost is defined by

$$V_\kappa(G) := \inf\left\{\mathcal{L}(c) : \exists \pi \in \hat{\mathcal{A}} = \mathbb{R}\times\mathbb{R}_+^N \times \mathcal{A} \text{ so that } \ Z_T^\pi(S) \geq G(S) \ \ \forall S \in \Omega\right\}.$$

For the second problem, we assume that a probability measure $\mathbb{P}$ on the canonical space Ω is given. Then, the corresponding problem is

$$V_\kappa^{\mathbb{P}}(G) := \inf\left\{\mathcal{L}(c) : \exists \pi \ \in \hat{\mathcal{A}} = \mathbb{R}\times\mathbb{R}_+^N \times \mathcal{A} \text{ so that } \ Z_T^\pi(S) \geq G(S) \ \ \mathbb{P}-\text{a.s}\right\}.$$

The main goal of this paper is to obtain the convex duality for these functionals and prove that they are equal if the measure $\mathbb{P}$ has conditional full support as defined in the next subsection.

2.6. **Main Results.** In order to formulate our results we need the following definitions. Recall that $\mathcal{C}^{++}[t,T]$ and the canonical space $\Omega = \mathcal{C}_1^{++}[t,T]$ are defined in subsection 2.1.

Definition 2.5. Consider the sample space $\hat{\Omega} := \Omega\times\mathcal{C}^{++}[0,T]$. Let $\hat{\mathbb{S}} = (\mathbb{S}^{(1)}, \mathbb{S}^{(2)})$ be the canonical process on $\hat{\Omega}$ and $\hat{\mathbb{F}}_t := \sigma(\hat{\mathbb{S}}_s,\, 0 \leq s \leq t)$ be the canonical filtration. A $(\kappa, \mathcal{L})$ *consistent price system* is a probability measure $\hat{\mathbb{Q}}$ on $\hat{\Omega}$ satisfying,

(1) $\mathbb{S}^{(2)}$ is a $\hat{\mathbb{Q}}$ martingale with respect to $\hat{\mathbb{F}}$;
(2) $(1-\kappa)\mathbb{S}_t^{(1)} \leq \mathbb{S}_t^{(2)} \leq (1+\kappa)\mathbb{S}_t^{(1)}$, $\hat{\mathbb{Q}}$–a.s.
(3) $\mathbb{E}_{\hat{\mathbb{Q}}}\left[f_i(\mathbb{S}^{(1)})\right] \leq \mathcal{L}_i$, for all $i = 1, \ldots, N$.

The set of all $(\kappa, \mathcal{L})$ consistent price systems is denoted by $\mathcal{M}_{\kappa,\mathcal{L}}$. □

Next we recall the notion of conditional full support. As usual, the support of a a probability measure $\mathbb{P}$ on a separable space, denoted by *supp* $\mathbb{P}$, is defined as the minimal closed set of full measure.

Definition 2.6. We say that a probability measure $\mathbb{P}$ has the *conditional full support property* if for all $t \in [0, T)$

$$\textit{supp}\ \mathbb{P}(\mathbb{S}_{|[t,T]}|\mathbb{F}_t) = \mathcal{C}^+_{\mathbb{S}_t}[t,T] \ \ \text{a.s.}$$

where $\mathbb{P}(\mathbb{S}_{|[t,T]}|\mathbb{F}_t)$ denotes the $\mathbb{F}_t$–conditional distribution of the $\mathcal{C}^+[t,T]$ valued random variable $\mathbb{S}_{|[t,T]}$ which is the restriction of the canonical process to $[t,T]$.

We are ready to state our main result.

Theorem 2.7. *Suppose Assumptions 2.1, 2.2, 2.3 hold. Assume* $0 < \kappa < 1/8$ *and let* $\mathbb{P}$ *be a probability measure which satisfies the conditional full support property. Then,*

$$V_\kappa^{\mathbb{P}}(G) = V_\kappa(G) = \sup_{\hat{\mathbb{Q}}\in\mathcal{M}_{\kappa,\mathcal{L}}} \mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})].$$

Clearly, $V^{\mathbb{P}}_\kappa(G)\le V_\kappa(G)$. Therefore, in order to prove Theorem 2.7 it suffices to prove the following two inequalities,

$$V^{\mathbb{P}}_\kappa(G)\ge \sup_{\hat{\mathbb{Q}}\in\mathcal{M}_{\kappa,\mathcal{L}}} \mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})] \tag{2.3}$$

and

$$V_\kappa(G)\le \sup_{\hat{\mathbb{Q}}\in\mathcal{M}_{\kappa,\mathcal{L}}} \mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})]. \tag{2.4}$$

The lower bound (2.3) is proved in Lemma 6.2 and the upper bound (2.4) is established in Lemma 6.3.

In the sequel, we always assume, without explicitly stating, that $0<\kappa<1/8$.

3. Reduction to Bounded Claims

The following result shows that in this market one can hedge certain claims in the tails with small cost. Similarly, to [12, 13], the proof is done by combining assumption (2.2) and the results of [1].

Lemma 3.1. *For any $K>0$ consider the European claim*

$$\alpha_K(S):=\frac{||S||}{K}+||S||\chi_{\{||S||\ge K\}}(S),\quad S\in\Omega,$$

where as before χ_A is the characteristic function. Under Assumption 2.2,

$$\lim_{K\to\infty} V_\kappa(\alpha_K)=0.$$

Proof. Let

$$\theta_0:=\theta_0(S)=0$$

and for a positive integer k we recursively define the stopping times by,

$$\theta_k:=\theta_k(S)=T\wedge\inf\{t>\theta_{k-1}:|S_t-S_{\theta_{k-1}}|=1\}.$$

Let $\mathbb{K}:=\mathbb{K}(S)=\min\{k:\theta_k=T\}$. Clearly, $\mathbb{K}<\infty$ for every $S\in\Omega$. By (2.2), it follows that there exists $c_q>1$ such that

$$q(x)\ge\frac{x^2}{c_q},\quad \forall x\ge c_q. \tag{3.1}$$

Consider the portfolio $\pi=(c,\gamma)$ where

$$\gamma_t=-\sum_{i=0}^{\mathbb{K}-1}\max_{0\le j\le i}S_{\theta_j}\ \chi_{(\theta_i,\theta_{i+1}]}(t),\qquad t\in[0,T],$$

and

$$c=(c_q^2,0,...,0,c_q),$$

i.e., we buy c_q many options $q(S_T)$ and invest in the riskless asset c_q^2 dollars. By summation by parts, Proposition 2.1 in Acciaio *et.al* [1] (see also Burkholder [7])

and (3.1), it follows that

$$\begin{aligned} Z_T^\pi(S) &= c_q^2 + c_q q(S_T) - \sum_{i=0}^{\mathbb{K}-1} \left[\max_{0\le j\le i} S_{\theta_j}\right] \ (S_{\theta_{i+1}} - S_{\theta_i}) \\ &\quad -\kappa \sum_{i=1}^{\mathbb{K}-1} S_{\theta_i} \left[\max_{0\le j\le i} S_{\theta_j} - \max_{0\le j\le i-1} S_{\theta_j}\right] \\ &\quad -\kappa S_0^2 \ - \ \kappa S_T \ \left[\max_{0\le j\le \mathbb{K}-1} S_{\theta_j}\right] \\ &\ge \ \frac{(1-8\kappa)}{4} \ \max_{0\le j\le \mathbb{K}} S_{\theta_j}^2. \end{aligned}$$

Observe that

$$||S|| \le 1 + \max_{0\le j\le \mathbb{K}} S_{\theta_j} \le 2 \max_{0\le j\le \mathbb{K}} S_{\theta_j}.$$

Also, since for any $S \in \Omega$, $S_0 = 1$, $\|S\| \ge 1$. Hence,

$$K\alpha_K(S) \le \|S\| + \|S\|^2 \le 2\|S\|^2 \le 8 \max_{0\le j\le \mathbb{K}} S_{\theta_j}^2.$$

Thus, (recall that $\kappa < \frac{1}{8}$)

$$Z_T^\pi(S) \ge \frac{(1-8\kappa)}{4} \ \left(\max_{0\le j\le \mathbb{K}} S_{\theta_j}^2\right) \ge \frac{K(1-8\kappa)}{32} \alpha_K(S).$$

We conclude that the super-replication cost of $[K(1-8\kappa)/32] \ \alpha_K$ is no more than the cost of this portfolio. Hence,

$$V_\kappa(\alpha_K) \le \frac{32}{(1-8\kappa)} \ \frac{c_q^2 + c_q \mathcal{L}_N}{K} \tag{3.2}$$

and the result follows after taking K to infinity. □

Next, we establish the reduction to bounded claims.

Lemma 3.2. *Under the assumptions of Theorem 2.7,, it sufficient to prove Theorem 2.7 for bounded claims.*

Proof. Let L be the Lispschitz constant in Assumption 2.1. For any $K \ge 1$ set

$$G_K(S) := G(S) \wedge [LK + G(0)], \qquad S \in \Omega.$$

From Assumption 2.1, it follows that $G(S) \le G(0) + L\|S\|$. Therefore, for all $K \ge 1$,

$$G(S) \le G_K(S) + (G(0) + L)\alpha_K(S).$$

Consequently,

$$V_\kappa(G) \le V_\kappa(G_K) + (G(0) + L)V_\kappa(\alpha_K), \quad V_\kappa^{\mathbb{P}}(G) \le V_\kappa^{\mathbb{P}}(G_K) + (G(0) + L)V_\kappa(\alpha_K).$$

Since G_K is bounded, if Theorem 2.7 holds for such claim, by the monotone convergence theorem we would have

$$V_\kappa(G) = \lim_{K\to\infty} V_\kappa(G_K) = \lim_{K\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}_{\kappa,\mathcal{L}}} \mathbb{E}_{\mathbb{Q}}[G_K(\mathbb{S}^{(1)})] = \sup_{\mathbb{Q}\in\mathcal{M}_{\kappa,\mathcal{L}}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S}^{(1)})].$$

Similar identities hold for $V_\kappa^{\mathbb{P}}(G)$ as well, proving the main theorem for all claims satisfying the Assumption 2.1. □

From now on, we will assume (without loss of generality) that there exists a constant $K > 0$ such that $0 \le G \le K$.

4. Lower Bound

In this section we establish estimates for the lower bound (2.3), under the assumptions of Theorem 2.7. We start with several definitions.

Recall that $\mathbb{D}[0,T]$ is the set of all *càdlàg*functions $f:[0,T]\to\mathbb{R}_+$. Denote by $\tilde{\mathbb{S}}_t$ the canonical process (i.e., $\tilde{\mathbb{S}}_t(\omega):=\omega_t$) on $\mathbb{D}[0,T]$. As usual, we consider the Borel σ–algebra with respect to the sup norm (this Borel σ–algebra coincides with the one generated by the Skorohod topology). Let $\tilde{\mathbb{F}}_t=\sigma\{\tilde{\mathbb{S}}_u|u\le t\}$ be the canonical filtration.

Let $\epsilon>0$, $n\in\mathbb{N}$ and $\mathcal{T}:=\{T_1,...,T_n,T\}$ be a partition of the interval $[0,T]$, i.e., $0<T_1<...<T_n<T$. In the sequel we shall always assume that $\epsilon<\ln(1+1/L)$ and $\epsilon<T_{i+1}-T_i$, $i=0,1,...,n-1$.

Definition 4.1. For any $0<\tilde{\kappa}<\kappa$, let $\mathcal{M}^{\mathcal{T},\epsilon}_{\tilde{\kappa},\mathcal{L}}$ be the set of all probability measures $\tilde{\mathbb{Q}}$ on the space $\mathbb{D}[0,T]$ satisfying,

(1) The canonical process $\tilde{\mathbb{S}}$ is of the form
$$\tilde{\mathbb{S}}_t=\sum_{i=0}^{n-1}\tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_k}\chi_{[\tilde{\tau}^{(\epsilon)}_k,\tilde{\tau}^{(\epsilon)}_{k+1})}+\tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_n}\chi_{[\tilde{\tau}^{(\epsilon)}_k,\tilde{\tau}^{(\epsilon)}_{n+1}]},$$
where $0=\tilde{\tau}^{(\epsilon)}_0\le\tilde{\tau}^{(\epsilon)}_1\le...\le\tilde{\tau}^{(\epsilon)}_{n+1}=T$ and $\tilde{\mathbb{S}}_0=1$.

(2) For any $k\le n$, on the event $\tilde{\tau}^{(\epsilon)}_{k+1}<T$ we have
$$|\ln\tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_{k+1}}-\ln\tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_k}|=\epsilon.$$

(3) For any $1\le k\le n+1$, $\tilde{\tau}^{(\epsilon)}_k\in\mathcal{T}$, $\tilde{\mathbb{Q}}$-a.s.

(4) There exists a $(\tilde{\mathbb{Q}},\tilde{\mathbb{F}})$ *càdlàg* martingale $\{\tilde{M}_t\}_{t=0}^{T}$ such that
$$(1-\tilde{\kappa})\tilde{\mathbb{S}}_t\le\tilde{M}_t\le(1+\tilde{\kappa})\tilde{\mathbb{S}}_t\quad\tilde{\mathbb{Q}}\text{-}a.s.;$$

(5) Finally,
$$\mathbb{E}_{\tilde{\mathbb{Q}}}[f_i(\tilde{\mathbb{S}})]\le\mathcal{L}_i-L\hat{C}(e^{4\epsilon}+\epsilon-1),\quad i=1,...,N-1,$$
$$E_{\tilde{\mathbb{Q}}}[f_N(\tilde{\mathbb{S}})]\le\frac{\mathcal{L}_N(1-L(e^{\epsilon}-1))-L\hat{C}(e^{\epsilon}-1)}{1+L(e^{\epsilon}-1)},$$
where $\hat{C}:=8\sqrt{c_q^2+c_q\mathcal{L}_N}$, and c_q is given in (3.1).

□

The following result provides a lower bound on the super–replication price $V^{\mathbb{P}}_{\kappa}(G)$.

Lemma 4.2. *Let $\mathbb{P}$ be a probability measure on Ω which satisfies the conditional full support property. Assume that*
$$\min\left(\frac{1+\kappa}{1+\tilde{\kappa}},\frac{1-\tilde{\kappa}}{1-\kappa}\right)\ge e^{2\epsilon}. \tag{4.1}$$
Then, for every partition $\mathcal{T}=\{T_1,\dots,T_n,T\}$,
$$V^{\mathbb{P}}_{\kappa}(G)\ge\sup_{\tilde{\mathbb{Q}}\in\mathcal{M}^{\mathcal{T},\epsilon}_{\tilde{\kappa},\mathcal{L}}}\mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{\mathbb{S}})]-L\hat{C}(e^{4\epsilon}+\epsilon-1).$$
We always use the standard convention that the supremum over the empty set is minus infinity.

Proof. Fix, $\epsilon > 0$ $\tilde{\kappa}$, $\mathcal{T}$ as above. If $\mathcal{M}^{\mathcal{T},\epsilon}_{\tilde{\kappa},\mathcal{L}} = \emptyset$ then the statement is trivial. Thus without loss of generality we assume that $\mathcal{M}^{\mathcal{T},\epsilon}_{\tilde{\kappa},\mathcal{L}} \neq \emptyset$. We fix an arbitrary measure $\tilde{\mathbb{Q}} \in \mathcal{M}^{\mathcal{T},\epsilon}_{\tilde{\kappa},\mathcal{L}}$ and we will show that

$$V^{\mathbb{P}}_{\kappa}(G) \geq \mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{\mathbb{S}})] - L\hat{C}(e^{4\epsilon} + \epsilon - 1). \tag{4.2}$$

The proof of the above inequality is completed in two steps. In the first step we use the conditional full support property of $\mathbb{P}$ and construct a consistent price system which is "close" to $\tilde{\mathbb{Q}}$. In the second step we use the super–replication property and the constructed consistent price system in order to obtain a lower bound on the price.

First step: In this step, we use the conditional full support property of $\mathbb{P}$ in a similar way to Guasoni, Rasonyi and Schachermayer [15].

Set $\tau^{(\epsilon)}_0 := \tau^{(\epsilon)}_0(\mathbb{S}) = 0$, and for any positive integer $k > 0$, recursively define

$$\tau^{(\epsilon)}_k := \tau^{(\epsilon)}_k(\mathbb{S}) = T \wedge \inf \left\{ t > \tau^{(\epsilon)}_{k-1} : |\ln \mathbb{S}_t - \ln \mathbb{S}_{\tau^{(\epsilon)}_{k-1}}| = \epsilon \right\}$$

where as before we denote by $\mathbb{S}$ the canonical process on Ω. Define a random integer by,

$$\mathbb{K} := \mathbb{K}(\mathbb{S}) = \min\{k : \tau^{(\epsilon)}_k = T\} - 1.$$

Then, it is clear that $0 \leq \mathbb{K} < \infty$. We also set,

$$S_k := \mathbb{S}_{\tau^{(\epsilon)}_{k \wedge \mathbb{K}}}, \quad 1 \leq k \leq n+1,$$

and

$$\sigma_k = \min\{t \in \mathcal{T} : t \geq \tau^{(\epsilon)}_k\}. \tag{4.3}$$

Recall that the positive integer n is the number of points in the fixed partition $\mathcal{T} = \{T_1, \ldots, T_n, T\}$.

For $\delta > 0$, $i = 1, ..., n$ and $j = \pm 1$, let $g^{i,j} : [0, T_i] \to \mathbb{R}_+$, be the linear functions satisfying

$$g^{i,j}_0 = 1, \quad \text{and} \quad g^{i,j}_{T_i} = e^{\epsilon j} + 2\delta j.$$

We assume that δ is sufficiently small so that $g^{i,j}$ is strictly positive. Next, on Ω we define the events

$$\begin{aligned} A^{(j)}_i &:= \{ \sup_{0 \leq t \leq T_i} |\mathbb{S}_t - g^{i,j}_t| < \delta\}, \qquad i = 1, ..., n,\ j = \pm 1 \\ A^{(0)}_T &:= \{ \sup_{0 \leq t \leq T} |\ \mathbb{S}_t - 1| < \delta\}. \end{aligned}$$

In view of the conditional full support property, all of these events have non-zero $\mathbb{P}$ probability. Also, observe that for sufficiently small δ, for $i = 1, \ldots, n$, $j = \pm 1$

$$A^{(j)}_i \subseteq B^{(j)}_i := \{\tau^{(\epsilon)}_1 \in [T_i - \epsilon/n, T_i],\ \mathbb{S}_{\tau^{(\epsilon)}_1} = \exp(\pm\epsilon)\}.$$

Also $A^{(0)}_T \subset B^{(0)}_T := \{\tau^{(\epsilon)}_1 = T\}$. Thus, we conclude that the events $B^{(0)}_T, B^{(j)}_i$, $i = 1, ..., n$, $j = \pm 1$ have non-zero $\mathbb{P}$ probabilities as well.

We proceed by induction. Assume that for a given $k \geq 1$ and any $j_1, ..., j_k = \pm 1$, $1 \leq i_1 < ... < i_k \leq n$, we have proved that the probability of the sets

$$B^{(j_1,\ldots,j_k)}_{i_1,\ldots,i_k} := \bigcap_{m=1}^{k} \left\{ \tau^{(\epsilon)}_m \in [T_{i_m} - \epsilon/n, T_{i_m}],\ \mathbb{S}_{\tau^{(\epsilon)}_m} = \exp(\epsilon \sum_{r=1}^{m} j_r) \right\}$$

and

$$B^{(j_1,\dots,j_{k-1},0)}_{i_1,\dots,i_{k-1},T} := \bigcap_{m=1}^{k-1} \left\{ \tau^{(\epsilon)}_m \in [T_{i_m} - \epsilon/n, T_{i_m}],\ \mathbb{S}_{\tau^{(\epsilon)}_m} = \exp(\epsilon \sum_{r=1}^{m} j_r) \right\} \bigcap \{\tau^{(\epsilon)}_k = T\}$$

have non-zero $\mathbb{P}$ probabilities.

Let $j_1, ..., j_{k+1} = \pm 1$, $1 \le i_1 < ... < i_{k+1} \le n$. On the event $\tau^{(\epsilon)}_k \le T_{i_k}$ define the random, linear function $g^{i_{k+1},j_{k+1}} : [\tau^{(\epsilon)}_k, T_{i_{k+1}}] \to \mathbb{R}_+$ by

$$g^{i_{k+1},j_{k+1}}_{\tau^{(\epsilon)}_k} = \exp(\epsilon \sum_{r=1}^{k} j_r) \ \text{ and } \ g^{i_{k+1},j_{k+1}}_{T_{i_{k+1}}} = \exp(\epsilon \sum_{r=1}^{k+1} j_r) + 2\delta j_{k+1}.$$

From the conditional full support property and Lemma 2.9 in Guasoni, Rasonyi and Schachermayer (2008), it follows that for any event $B \in \mathbb{F}_{\tau^{(\epsilon)}_k}$ the conditional probabilities

$$\mathbb{P}\left(\sup_{\tau^{(\epsilon)}_k \le t \le T_{i_{k+1}}} | \, \mathbb{S}_t - g^{i_{k+1},j_{k+1}}_t \, | < \delta \ | \ B^{(j_1,\dots,j_k)}_{i_1,\dots,i_k} \cap B \right) > 0,$$

and

$$\mathbb{P}\left(\sup_{\tau^{(\epsilon)}_k \le t \le T} | \, \mathbb{S}_t - \exp(\epsilon \sum_{r=1}^{k} j_r) \, | < \delta \ | \ B^{(j_1,\dots,j_k)}_{i_1,\dots,i_k} \cap B \right) > 0,$$

provided that $\mathbb{P}(B^{(j_1,\dots,j_k)}_{i_1,\dots,i_k} \cap B) > 0$. Thus, similarly to the case $k = 1$, for sufficiently small δ we conclude that the $\mathbb{P}$ probabilities of the following events

$$B^{(j_1,\dots,j_{k+1})}_{i_1,\dots,i_{k+1}} := \bigcap_{m=1}^{k+1} \left\{ \tau^{(\epsilon)}_m \in [T_{i_m} - \epsilon/n, T_{i_m}],\ \mathbb{S}_{\tau^{(\epsilon)}_m} = \exp(\epsilon \sum_{r=1}^{m} j_r) \right\}$$

and

$$B^{(j_1,\dots,j_k,0)}_{i_1,\dots,i_k,T} := \bigcap_{m=1}^{k} \left\{ \tau^{(\epsilon)}_m \in [T_{i_m} - \epsilon/n, T_{i_m}],\ \mathbb{S}_{\tau^{(\epsilon)}_m} = \exp(\epsilon \sum_{r=1}^{m} j_r) \right\} \bigcap \{\tau^{(\epsilon)}_{k+1} = T\}$$

are positive. This holds true for any $k \le n+1$.

Recall the measure $\tilde{\mathbb{Q}} \in \mathcal{M}^{\mathcal{T},\epsilon}_{\tilde{\kappa},\mathcal{L}}$ that was fixed at the start of the proof and the σ_k's defined by (4.3). In view of the above discussion, and by using similar arguments as in Lemma 2.4 in Guasoni, Rasonyi and Schachermayer (2008), it follows that there exists another probability measure $\hat{\mathbb{Q}} \ll \mathbb{P}$ such that the distribution of $(S_1, ..., S_{n+1}, \sigma_1, ..., \sigma_{n+1})$ under $\hat{\mathbb{Q}}$ is equal to the distribution of $(\tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_1}, \dots, \tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_{n+1}}, \tilde{\tau}^{(\epsilon)}_1, ..., \tilde{\tau}^{(\epsilon)}_{n+1})$ under $\tilde{\mathbb{Q}}$, and in addition for any $i \le n$, we have

$$\hat{\mathbb{Q}}(\mathbb{S}_{i+1}, \sigma_{i+1} | \mathbb{F}_{\tau^{(\epsilon)}_i}) = \hat{\mathbb{Q}}(\mathbb{S}_{i+1}, \sigma_{i+1} | \mathbb{S}_1, ..., \mathbb{S}_i, \sigma_1, ..., \sigma_i), \ \hat{\mathbb{Q}} \text{ a.s.} \tag{4.4}$$

Also observe that from our construction it follows that for any k,

$$|\sigma_k - \tau^{(\epsilon)}_k| \le \frac{\epsilon}{n}, \quad \hat{\mathbb{Q}} \ \text{ a.s.} \tag{4.5}$$

and

$$S_{k+1} e^{-2\epsilon} \le \mathbb{S}_t \le S_{k+1} e^{2\epsilon}, \quad \forall t \in [\tau^{(\epsilon)}_k, \tau^{(\epsilon)}_{k+1}] \ \ \hat{\mathbb{Q}} \ \text{ a.s.} \tag{4.6}$$

Now, we arrive to the second step of the proof.

Second step: Since $\tilde{\mathbb{Q}} \in \mathcal{M}^{\mathcal{T},\epsilon}_{\tilde{\kappa},\mathcal{L}}$, the definition of this set implies that there exists an associated martingale $\{\tilde{M}_t\}_{t=0}^{T}$ which satisfies

$$(1-\tilde{\kappa})\tilde{\mathbb{S}}_t \le \tilde{M}_t \le (1+\tilde{\kappa})\tilde{\mathbb{S}}_t, \quad t \in [0,T] \ \ \tilde{\mathbb{Q}} \ \text{ a.s.}$$

Then, for any $k \le n+1$ there exists a measurable function

$$\psi_k : \mathbb{R}^k \times \mathcal{T} \to \mathbb{R}_+$$

such that

$$\tilde{M}_{\tilde{\tau}_k^{(\epsilon)}} = \psi_k(\tilde{\mathbb{S}}_{\tilde{\tau}_1^{(\epsilon)}}, \ldots, \tilde{\mathbb{S}}_{\tilde{\tau}_k^{(\epsilon)}}, \tilde{\tau}_1^{(\epsilon)}, ..., \tilde{\tau}_k^{(\epsilon)}).$$

Moreover,

$$(1-\tilde{\kappa})\tilde{\mathbb{S}}_{\tilde{\tau}_k^{(\epsilon)}} \le \tilde{M}_{\tilde{\tau}_k^{(\epsilon)}} \le (1+\tilde{\kappa})\tilde{\mathbb{S}}_{\tilde{\tau}_k^{(\epsilon)}}, \quad k \le n+1 \ \ \tilde{\mathbb{Q}} \ \text{ a.s.} \tag{4.7}$$

Then, on Ω we define the stochastic process M simply by

$$M_k = \psi_k(S_1, ..., S_k, \sigma_1, ..., \sigma_k).$$

In view of (4.4) anf (4.7), it follows that for any k,

$$\mathbb{E}_{\hat{\mathbb{Q}}}(M_{k+1} \mid \mathbb{F}_{\tau_k^{(\epsilon)}}) = M_k \tag{4.8}$$

and

$$(1-\tilde{\kappa})S_k \le M_k \le (1+\tilde{\kappa})S_k \ \ \hat{\mathbb{Q}} \ \text{ a.s.} \tag{4.9}$$

Now, let $\pi = (c, \gamma)$ be a $\mathbb{P}$ almost-surely super-replicating portfolio. By (4.1), (4.6)–(4.9) and by summation by parts, it follows that

$$\begin{aligned}
&\mathbb{E}_{\tilde{\mathbb{Q}}}\left(\gamma_T \mathbb{S}_T - \kappa|\gamma_T|\mathbb{S}_T + (1-\kappa)\int_{[0,T]} \mathbb{S}_u d\gamma_u^- - (1+\kappa)\int_{[0,T]} \mathbb{S}_u d\gamma_u^+\right) \\
&\le \mathbb{E}_{\hat{\mathbb{Q}}}\left(\gamma_T M_{n+1} + (1-\tilde{\kappa})\sum_{k=0}^{n} S_{k+1}\int_{[\tau_k^{(\epsilon)}, \tau_{k+1}^{(\epsilon)}]} d\gamma_u^-\right) \\
&\quad -\mathbb{E}_{\hat{\mathbb{Q}}}\left((1+\tilde{\kappa})\sum_{k=0}^{n} S_{k+1}\int_{[\tau_k^{(\epsilon)}, \tau_{k+1}^{(\epsilon)}]} d\gamma_u^+\right) \\
&\le \mathbb{E}_{\hat{\mathbb{Q}}}\left(\gamma_T M_{n+1} + \sum_{k=0}^{n} M_{k+1}\left(\int_{[\tau_k^{(\epsilon)}, \tau_{k+1}^{(\epsilon)}]} d\gamma_u^- - \int_{[\tau_k^{(\epsilon)}, \tau_{k+1}^{(\epsilon)}]} d\gamma_u^+\right)\right) \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}(\sum_{k=1}^{n} \gamma_{\tau_k^{(\epsilon)}}(M_{k+1} - M_k)) = 0.
\end{aligned} \tag{4.10}$$

Next, we introduce the stochastic process $\{\tilde{S}_t\}_{t=0}^{T}$ by,

$$\tilde{S}_t := \sum_{k=0}^{n-1} S_k \chi_{[\sigma_k, \sigma_{k+1})}(t) + S_n \chi_{[\sigma_n, T]}(t),$$

where we set $\sigma_0 = 0$. From our construction, it follows that the distribution (on the space $\mathbb{D}[0,T]$) of $\{\tilde{S}_t\}_{t=0}^{T}$ under $\hat{\mathbb{Q}}$ is equal to the distribution of $\tilde{\mathbb{S}}$ under $\tilde{\mathbb{Q}}$. Thus,

$$\mathbb{E}_{\hat{\mathbb{Q}}} G(\tilde{S}) = \mathbb{E}_{\tilde{\mathbb{Q}}} G(\tilde{\mathbb{S}}) \ \text{ and } \ \mathbb{E}_{\hat{\mathbb{Q}}} f_i(\tilde{S}) = \mathbb{E}_{\tilde{\mathbb{Q}}} f_i(\tilde{\mathbb{S}}), \quad i \le N. \tag{4.11}$$

We next use the Assumption 2.1 and the properties (4.5)–(4.6). The result is the following inequalities that hold $\hat{\mathbb{Q}}$ a.s.,

$$
\begin{aligned}
|G(\tilde{S}) - G(\mathbb{S})| &\leq L(e^{4\epsilon} + \epsilon - 1)\|\tilde{S}\|, \\
|f_i(\tilde{S}) - f_i(\mathbb{S})| &\leq L(e^{4\epsilon} + \epsilon - 1)\|\tilde{S}\|, \qquad \text{for } \ i \leq N-1.
\end{aligned} \tag{4.12}
$$

From Assumption 2.2 it follows that (recall that $e^{\epsilon} < \frac{L+1}{L}$) for any positive real numbers x, y

$$
|\ln x - \ln y| \leq \epsilon \Rightarrow q(y) \leq \frac{q(x)(1 + L(e^{\epsilon}-1)) + L(e^{\epsilon}-1)x}{1 - L(e^{\epsilon}-1)}.
$$

We conclude that

$$
f_N(\mathbb{S}) \leq \frac{f_N(\tilde{S})(1 + L(e^{\epsilon}-1)) + L(e^{\epsilon}-1)||\tilde{S}||}{1 - L(e^{\epsilon}-1)} \quad \hat{\mathbb{Q}} \ \text{ a.s.} \tag{4.13}
$$

From (3.1), Assumption 2.2 and the Doob inequality, it follows that

$$
\begin{aligned}
\mathbb{E}_{\hat{\mathbb{Q}}}[\|\tilde{S}\|^2] &= \mathbb{E}_{\tilde{\mathbb{Q}}}[\|\tilde{\mathbb{S}}\|^2] \leq 4\mathbb{E}_{\tilde{\mathbb{Q}}}[\|\tilde{M}\|^2] \leq 16\mathbb{E}_{\tilde{\mathbb{Q}}}[\tilde{M}_T^2] \\
&\leq 64\mathbb{E}_{\tilde{\mathbb{Q}}}[\tilde{\mathbb{S}}_T^2] \leq 64[c_q^2 + c_q\mathcal{L}_N] = \hat{C}^2,
\end{aligned}
$$

where the constants $\hat{C}$ and c_q are as in Definition 4.1. Also, the Holder inequality yields that

$$
\mathbb{E}_{\hat{\mathbb{Q}}}[\|\tilde{S}\|] \leq \hat{C}. \tag{4.14}
$$

Finally (4.11)–(4.14) and the fact that $\tilde{\mathbb{Q}} \in \mathcal{M}^{\mathcal{T},\epsilon}_{\tilde{\kappa},\mathcal{L}}$ implies that $\mathbb{E}_{\hat{\mathbb{Q}}} f_i(\mathbb{S}) \leq \mathcal{L}_i$, for every $i \leq N$. Therefore, using (4.10)–(4.14) and the relation $\hat{\mathbb{Q}} \ll \mathbb{P}$, we arrive at

$$
\mathcal{L}(c) \geq \mathbb{E}_{\hat{\mathbb{Q}}}[c \cdot f(\mathbb{S})] \geq \mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S})] \geq \mathbb{E}_{\tilde{\mathbb{Q}}} G[(\tilde{\mathbb{S}})] - L\hat{C}(e^{4\epsilon} + \epsilon - 1).
$$

Since the above inequality holds for every $\mathbb{P}$ almost-surely super-replicating strategy $\pi = (c, \gamma)$, this proves the inequality (4.2) and completes the proof of this lemma. □

5. Estimates for the Upper Bound

In this section we establish estimates that will be used in the proof of the upper bound, under the assumptions of Theorem 2.7.

We fix $\epsilon \in (0, \ln(1 + 1/L))$ and start with two definitions.

Definition 5.1. A function $F \in \mathbb{D}[0,T]$ belongs to $\mathbb{D}^{(\epsilon)}$, if it satisfies the followings,

(1) $F_0 = 1$.
(2) F is piecewise constant with jumps at times $t_1, ..., t_n$, where

$$
t_0 = 0 < t_1 < t_2 < ... < t_n < T.
$$

(3) For any $k = 1, ..., n$, $|\ln F_{t_k} - \ln F_{t_{k-1}}| = \epsilon$.
(4) For any $k = 1, ..., n$, $t_k - t_{k-1} \in U^{(\epsilon)}_k$, where

$$
U^{(\epsilon)}_k := \left\{ i\epsilon/(2^k) : i = 1, 2, \ldots, \right\} \cup \left\{ \epsilon/(i2^k) : i = 1, 2, \ldots, \right\},
$$

are the sets of possible differences between two consecutive jump times. We emphasise, in the fourth condition, the dependence of the set $U^{(\epsilon)}_k$ on k. So as k gets larger, jump times take values in a finer grid.

□

Definition 5.2. For $\tilde{\kappa}, \Lambda > 0$, let $\mathcal{M}^{\epsilon,\Lambda}_{\tilde{\kappa},\mathcal{L}}$ be the set of all probability measures $\tilde{\mathbb{Q}}$ on the space $\mathbb{D}[0,T]$ such that the following holds,

(1) The probability measure $\tilde{\mathbb{Q}}$ is supported on the set $\mathbb{D}^{(\epsilon)}$.

(2) There exists a *càdlàg* $(\tilde{\mathbb{Q}}, \tilde{\mathbb{F}})$ martingale $\{\tilde{M}_t\}_{t=0}^{T}$ such that

$$(1-\tilde{\kappa})\tilde{\mathbb{S}}_t \le \tilde{M}_t \le (1+\tilde{\kappa})\tilde{\mathbb{S}}_t \quad \tilde{\mathbb{Q}} \quad a.s.$$

(3) Let $\hat{C}$ be as in Definition 4.1 and L be as in Assumption 2.1. Set

$$B := L(e^{2\epsilon}+\epsilon-1)\frac{\hat{C}^2}{2(1-8\kappa)} + 2L(e^{\epsilon}-1)\mathcal{L}_N + \epsilon$$

For any $i < N$,

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[f_i(\tilde{\mathbb{S}})] \le \mathcal{L}_i + B,$$

and

$$\mathbb{E}_{\tilde{\mathbb{Q}}}[f_N(\tilde{\mathbb{S}}) \wedge \Lambda(\tilde{\mathbb{S}}_T+1)] \le \mathcal{L}_N + B.$$

□

The following result provides an upper bound on the model–free super–replication price $V_\kappa(G)$.

Lemma 5.3. *Assume that*

$$\min\left(\frac{1+\tilde{\kappa}}{1+\kappa}, \frac{1-\kappa}{1-\tilde{\kappa}}\right) \ge e^{4\epsilon}. \tag{5.1}$$

Then

$$V_\kappa(G) \le \left(\sup_{\tilde{\mathbb{Q}}\in\mathcal{M}^{\epsilon,\Lambda}_{\tilde{\kappa},\mathcal{L}}} \mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{\mathbb{S}})]\right)^+ + L(e^{2\epsilon}+\epsilon-1)\frac{\hat{C}^2}{2(1-8\kappa)}.$$

Again, we use the standard convention that the supremum over the empty set is minus infinity. In particular, if $\mathcal{M}^{\epsilon,\Lambda}_{\tilde{\kappa},\mathcal{L}}$ is empty, then the above lemma states that $V_\kappa(G) \le L(e^{2\epsilon}+\epsilon-1)\frac{\hat{C}^2}{2(1-8\kappa)}$.

Proof. The proof is completed in two steps. In the first step, we apply the results that deal with the "classical" super–replication with proportional transaction costs.

First step: Since $\mathbb{D}^{(\epsilon)}$ is countable, there exists a probability measure $\tilde{\mathbb{P}}$ satisfying $\tilde{\mathbb{P}}(\mathbb{D}^{(\epsilon)}) = 1$ and $\tilde{\mathbb{P}}(\{F\}) > 0$ for all $F \in \mathbb{D}^{(\epsilon)}$. Consider the filtered probability space $(\mathbb{D}[0,T], \{\tilde{\mathbb{F}}_t\}_{t=0}^{T}, \tilde{\mathbb{F}}_T, \tilde{\mathbb{P}})$. Denote by $\mathcal{M}_{\tilde{\kappa}}$ the set of all consistent price systems in $\mathbb{D}^{(\epsilon)}$. Namely, $\tilde{\mathbb{Q}} \in \mathcal{M}_{\tilde{\kappa}}$ if $\tilde{\mathbb{Q}}$ is equivalent to $\tilde{\mathbb{P}}$ and there exists a *càdlàg* martingale $\{\tilde{M}_t\}_{t=0}^{T}$ (with respect to $\tilde{\mathbb{Q}}$ and $\tilde{\mathbb{F}}$) such that

$$(1-\tilde{\kappa})\tilde{\mathbb{S}}_t \le \tilde{M}_t \le (1+\tilde{\kappa})\tilde{\mathbb{S}}_t \quad \tilde{\mathbb{P}} \text{ a.s.}$$

Let $X := X(\tilde{\mathbb{S}})$ be random variable which is $\tilde{\mathbb{F}}_T$ measurable and bounded from below by a multiple of $1+\tilde{\mathbb{S}}_T$. Set

$$c_0 := \sup_{\tilde{\mathbb{Q}}\in\mathcal{M}_{\tilde{\kappa}}} \mathbb{E}_{\tilde{\mathbb{Q}}}[X]. \tag{5.2}$$

From Theorem 1.5 in Schachermayer [21], it follows that there exists a predictable stochastic process of bounded variation $\{\tilde{\gamma}_t\}_{t=0}^T$ such that $\tilde{\gamma}_0 = \tilde{\gamma}_T = 0$ and

$$c_0 + (1-\tilde{\kappa})\int_{[0,T]} \tilde{\mathbb{S}}_u d\tilde{\gamma}_u^- - (1+\tilde{\kappa})\int_{[0,T]} \tilde{\mathbb{S}}_u d\tilde{\gamma}_u^+ \geq X, \quad \tilde{\mathbb{P}} \ \text{ a.s.}$$

Thus, there exists a predictable map $\tilde{\gamma} : \mathbb{D}^{(\epsilon)} \to \mathbb{L}^\infty[0,T]$ such that for any $F \in \mathbb{D}^{(\epsilon)}$ $\tilde{\gamma}_0(F) = \tilde{\gamma}_T(F) = 0$ and

$$c_0 + (1-\tilde{\kappa})\int_{[0,T]} F_u d\tilde{\gamma}_u^-(F) - (1+\tilde{\kappa})\int_{[0,T]} F_u d\tilde{\gamma}_u^+(F) \geq X(F), \tag{5.3}$$

where $\mathbb{L}^\infty[0,T]$ is the set of all bounded functions on the interval $[0,T]$. Next, choose $(c_1,...,c_N) \in \mathbb{R}_+^N$ and consider the random variable

$$X = X(\tilde{\mathbb{S}}) = G(\tilde{\mathbb{S}}) - \sum_{i=1}^{N-1} c_i f_i(\tilde{\mathbb{S}}) - c_N(f_N(\tilde{\mathbb{S}}) \wedge \Lambda(\tilde{\mathbb{S}}_T + 1)).$$

Recall, that in Assumption 2.2 we assumed that if f_i is path dependent then it is bounded. This together with the Lipschitz continuity of f_i, $i = 1,...,N-1$ yields that $f_1(\tilde{\mathbb{S}}),...,f_{N-1}(\tilde{\mathbb{S}})$ are bounded by a multiple of $1+\tilde{\mathbb{S}}_T$, and so X is bounded by a multiple of $1+\tilde{\mathbb{S}}_T$ as well.

Let $(c_0,\tilde{\gamma})$ be such that (5.2) and (5.3) hold true.

Next, we lift the trading strategy $\tilde{\gamma}$ to a trading strategy on the space Ω. We start with some preparations. Recall the definition of the stopping times $\tau_k^{(\epsilon)} := \tau_k^{(\epsilon)}(\mathbb{S})$, $k \geq 0$, and $\mathbb{K} := \mathbb{K}(\mathbb{S}) = \min\{k : \tau_k^{(\epsilon)} = T\} - 1$.

Set,

$$\begin{aligned} \hat{\tau}_k^{(\epsilon)} &:= \sum_{i=1}^k \Delta\hat{\tau}_i^{(\epsilon)}, \quad \text{where} \\ \Delta\hat{\tau}_i^{(\epsilon)} &= \max\{\Delta t \in U_i^{(\epsilon)} : \Delta t < \Delta\tau_i^{(\epsilon)} := \tau_i^{(\epsilon)} - \tau_{i-1}^{(\epsilon)}\}. \end{aligned}$$

It is clear that $0 = \hat{\tau}_0^{(\epsilon)} < \hat{\tau}_1^{(\epsilon)} < ... < \hat{\tau}_{\mathbb{K}}^{(\epsilon)} < T$ and $\hat{\tau}_k^{(\epsilon)} < \tau_k^{(\epsilon)}$ for all $k = 0,\dots,\mathbb{K}$.

We now define $\Psi : \Omega \to \mathbb{D}^{(\epsilon)}$ by

$$\Psi_t(\mathbb{S}) := \sum_{k=0}^{\mathbb{K}-1} \mathbb{S}_{\tau_k^{(\epsilon)}} \chi_{[\hat{\tau}_k^{(\epsilon)},\hat{\tau}_{k+1}^{(\epsilon)})}(t) + \mathbb{S}_{\tau_{\mathbb{K}}^{(\epsilon)}} \chi_{[\hat{\tau}_{\mathbb{K}}^{(\epsilon)},T]}(t).$$

Finally, define the hedge $\pi = (c,\gamma)$ where $c = (c_0,c_1,...,c_N)$ and

$$\gamma(\mathbb{S}) := \sum_{k=1}^{\mathbb{K}} \tilde{\gamma}_{\hat{\tau}_k^{(\epsilon)}}(\Psi(\mathbb{S}))\chi_{(\tau_k^{(\epsilon)},\tau_{k+1}^{(\epsilon)}]}(t).$$

We continue by estimating the portfolio value $Z_T^\pi(\mathbb{S})$. Set

$$\begin{aligned} I \ &:= \ I(\mathbb{S}) = \gamma_T\mathbb{S}_T - \kappa|\gamma_T|\mathbb{S}_T + (1-\kappa)\int_{[0,T]} \mathbb{S}_u d\gamma_u^- - (1+\kappa)\int_{[0,T]} \mathbb{S}_u d\gamma_u^+ \\ &\quad -(1-\tilde{\kappa})\int_{[0,T]} \Psi_u(\mathbb{S}) d\tilde{\gamma}_u^-(\Psi(\mathbb{S})) + (1+\tilde{\kappa})\int_{[0,T]} \Psi_u(\mathbb{S}) d\tilde{\gamma}_u^+(\Psi(\mathbb{S})). \end{aligned}$$

From Assumption 2.2 it follows that for any $x,y > 0$

$$|\ln x - \ln y| < \epsilon \Rightarrow q(x) \geq \frac{(1-L(e^\epsilon-1))q(y) - L(e^\epsilon-1)y}{1+L(e^\epsilon-1)}.$$

Thus, from Assumptions 2.1, 2.2 and (5.3), it follows that

$$
\begin{aligned}
(5.4)\; Z^{\pi}_T(\mathbb{S}) - G(\mathbb{S}) &\geq I - (G(\mathbb{S}) - G(\Psi(\mathbb{S}))) - \sum_{i=1}^{N} c_i (f_i(\Psi(\mathbb{S})) - f_i(\mathbb{S})) \\
&\geq I - L\left(1 + \sum_{i=1}^{N-1} c_i\right)\left(e^{2\epsilon} + \sum_{j=1}^{\infty} \epsilon 2^{-j} - 1\right)\|\mathbb{S}\| - Lc_N(e^{\epsilon}-1)\frac{2f_N(\mathbb{S}) + \|S\|}{1 + L(e^{\epsilon}-1)} \\
&\geq I - L\left(1 + \sum_{i=1}^{N-1} c_i\right)\left(e^{2\epsilon} + \epsilon - 1\right)\|\mathbb{S}\| - Lc_N(e^{\epsilon}-1)\left(2f_N(\mathbb{S}) + \|S\|\right).
\end{aligned}
$$

It remains to estimate the term I. To simplify the calculations, we use the notation $\gamma = \gamma(\mathbb{S})$ and $\tilde{\gamma} = \tilde{\gamma}(\Psi(\mathbb{S}))$. Then, in view of (5.1),

$$
\begin{aligned}
&\gamma_T \mathbb{S}_T - \kappa|\gamma_T|\mathbb{S}_T + (1-\kappa)\int_{[0,T]} \mathbb{S}_u d\gamma^-_u - (1+\kappa)\int_{[0,T]} \mathbb{S}_u d\gamma^+_u \\
&\quad \geq \gamma_T \mathbb{S}_T - \kappa|\gamma_T|\mathbb{S}_T + \sum_{k=1}^{\mathbb{K}} \mathbb{S}_{\tau^{(\epsilon)}_{k-1}} \int_{[\tau^{(\epsilon)}_k, \tau^{(\epsilon)}_{k+1}]} [(1-\tilde{\kappa})d\gamma^-_u - (1+\tilde{\kappa})d\gamma^+_u] \\
&\quad = \gamma_T \mathbb{S}_T - \kappa|\gamma_T|\mathbb{S}_T + \sum_{k=1}^{\mathbb{K}} \mathbb{S}_{\tau^{(\epsilon)}_{k-1}} \int_{[\tau^{(\epsilon)}_k, \tau^{(\epsilon)}_{k+1}]} [-d\gamma_u - \tilde{\kappa}|d\gamma_u|] \\
&\quad \geq \gamma_T \mathbb{S}_T - \kappa|\gamma_T|\mathbb{S}_T + \sum_{k=0}^{\mathbb{K}-1} \Psi_{\hat{\tau}^{(\epsilon)}_k}(\mathbb{S}) \int_{[\hat{\tau}^{(\epsilon)}_k, \hat{\tau}^{(\epsilon)}_{k+1}]} [-d\tilde{\gamma}_u - \tilde{\kappa}|d\tilde{\gamma}_u|] \\
&\quad = \gamma_T \mathbb{S}_T - \kappa|\gamma_T|\mathbb{S}_T + (1-\tilde{\kappa})\int_{[0,\hat{\tau}^{(\epsilon)}_{\mathbb{K}}]} \Psi_u(\mathbb{S}) d\tilde{\gamma}^-_u - (1+\tilde{\kappa})\int_{[0,\hat{\tau}^{(\epsilon)}_{\mathbb{K}}]} \Psi_u(\mathbb{S}) d\tilde{\gamma}^+_u \\
&\quad \geq (1-\tilde{\kappa})\int_{[0,T]} \Psi_u(\mathbb{S}) d\tilde{\gamma}^-_u - (1+\tilde{\kappa})\int_{[0,T]} \Psi_u(\mathbb{S}) d\tilde{\gamma}^+_u.
\end{aligned}
$$

Hence, we conclude that $I \geq 0$. We use this inequality together with (3.2) and (5.4). The result is,

$$
\begin{aligned}
V_\kappa(G) &\leq \mathcal{L}(c) + L(e^{2\epsilon} + \epsilon - 1)(1 + \sum_{i=1}^{N} c_i) V_\kappa(\|\mathbb{S}\|) + 2L(e^{\epsilon}-1)c_N V_\kappa(f_N(\mathbb{S})) \\
&\leq \mathcal{L}(c) + L(e^{2\epsilon} + \epsilon - 1)\frac{\hat{C}^2}{2(1-8\kappa)}(1 + \sum_{i=1}^{N} c_i) + 2L(e^{\epsilon}-1)c_N \mathcal{L}_N.
\end{aligned}
$$

This together with (5.2) yields

$$
(5.5) \quad V_\kappa(G) \leq \inf_{c_1,\ldots,c_N \geq 0} \sup_{\tilde{\mathbb{Q}} \in \mathcal{M}_{\tilde{\kappa}}} \left(\mathbb{E}_{\tilde{\mathbb{Q}}}[\ \xi\ ] + \sum_{i=1}^{N} c_i A_i\right) + L(e^{2\epsilon} + \epsilon - 1)\frac{\hat{C}^2}{2(1-8\kappa)},
$$

where

$$
\begin{aligned}
\xi &:= G(\tilde{\mathbb{S}}) - \sum_{i=1}^{N-1} c_i f_i(\tilde{\mathbb{S}}) - c_N(f_N(\tilde{\mathbb{S}}) \wedge \Lambda(\tilde{\mathbb{S}}_T + 1)), \\
A_i &:= \mathcal{L}_i + L(e^{2\epsilon} + \epsilon - 1)\frac{\hat{C}^2}{2(1-8\kappa)} + 2L(e^{\epsilon}-1)\mathcal{L}_N = \mathcal{L}_i + B - \epsilon, \quad i \leq N.
\end{aligned}
$$

Second Step: The next step is to interchange the order of the infimum and supremum in (5.5). Consider the compact set $H := [0, K/\epsilon]^N$, where recall K is satisfying $G \leq K$. Define the function $\mathcal{G} : H \times \mathcal{M}_{\tilde{\kappa}} \to \mathbb{R}$ by

$$\mathcal{G}(h, \tilde{\mathbb{Q}}) = \mathbb{E}_{\tilde{\mathbb{Q}}}\left[G(\tilde{\mathbb{S}}) - \sum_{i=1}^{N-1} h_i f_i(\tilde{\mathbb{S}}) - h_N(f_N(\tilde{\mathbb{S}}) \wedge \Lambda(\tilde{\mathbb{S}}_T + 1))\right] + \sum_{i=1}^{N} h_i A_i,$$

where $h = (h_1, ..., h_N)$. Notice that $\mathcal{G}$ is affine in each of the variables, and continuous in the first variable. The set $\mathcal{M}_{\tilde{\kappa}}$ can be naturally considered as a subset of the vector space $\mathbb{R}^{\mathbb{D}^{(\epsilon)}}$. Let us show that $\mathcal{M}_{\tilde{\kappa}}$ is a convex set. Let $\tilde{\mathbb{Q}}_1, \tilde{\mathbb{Q}}_2 \in \mathcal{M}_{\tilde{\kappa}}$ and let $\lambda \in (0,1)$. Consider the measure $\tilde{\mathbb{Q}} = \lambda\tilde{\mathbb{Q}}_1 + (1-\lambda)\tilde{\mathbb{Q}}_2$. For $i = 1, 2$ let $\{\tilde{M}^{(i)}_t\}_{t=0}^{T}$ be a martingale with respect to $\tilde{\mathbb{Q}}_i$ and $\tilde{\mathbb{F}}$, such that

$$(1-\tilde{\kappa})\tilde{\mathbb{S}}_t \leq \tilde{M}^{(i)}_t \leq (1+\tilde{\kappa})\tilde{\mathbb{S}}_t \ \ \tilde{\mathbb{P}} \ \text{ a.s.}$$

Define the stochastic process

$$\tilde{M}_t = \lambda \tilde{M}^{(1)}_t \ \left[\frac{d\tilde{\mathbb{Q}}_1}{d\tilde{\mathbb{Q}}}|\tilde{\mathbb{F}}_t\right] + (1-\lambda)\tilde{M}^{(2)}_t \ \left[\frac{d\tilde{\mathbb{Q}}_2}{d\tilde{\mathbb{Q}}}|\tilde{\mathbb{F}}_t\right], \quad t \in [0, T].$$

Clearly, $\{\tilde{M}_t\}_{t=0}^{T}$ is a martingale with respect to $\tilde{\mathbb{Q}}$ and $\tilde{\mathbb{F}}$. Also, since $\tilde{M}_t$ is a (random) convex combination of $\tilde{M}^{(1)}_t$ and $\tilde{M}^{(2)}_t$,

$$(1-\tilde{\kappa})\tilde{\mathbb{S}}_t \leq \tilde{M}_t \leq (1+\tilde{\kappa})\tilde{\mathbb{S}}_t \ \ \tilde{\mathbb{P}} \ \text{ a.s.}$$

Hence, $\tilde{\mathbb{Q}} \in \mathcal{M}_{\tilde{\kappa}},$. This proves that $\mathcal{M}_{\tilde{\kappa}}$ is a convex set. Next, we apply the min–max theorem, Theorem 2, in Beiglböck, Henry-Labordère and Penkner [3] to $\mathcal{G}$. The result is,

$$\inf_{h \in H} \sup_{\tilde{\mathbb{Q}} \in \mathcal{M}_{\tilde{\kappa}}} \mathcal{G}(h, \tilde{\mathbb{Q}}) = \sup_{\tilde{\mathbb{Q}} \in \mathcal{M}_{\tilde{\kappa}}} \inf_{h \in H} \mathcal{G}(h, \tilde{\mathbb{Q}}) \leq \sup_{\tilde{\mathbb{Q}} \in \mathcal{M}_{\tilde{\kappa}}} \mathcal{G}(h^{\tilde{\mathbb{Q}}}, \tilde{\mathbb{Q}}),$$

where

$$h^{\tilde{\mathbb{Q}}}_i = \frac{K}{\epsilon}\chi_{\left\{\mathbb{E}_{\tilde{\mathbb{Q}}}[f_i(\tilde{\mathbb{S}})] \geq \mathcal{L}_i + B\right\}}, \quad i \leq N-1, \quad h^{\tilde{\mathbb{Q}}}_N = \frac{K}{\epsilon}\chi_{\left\{\mathbb{E}_{\tilde{\mathbb{Q}}}[f_N(\tilde{\mathbb{S}}) \wedge \Lambda(\tilde{\mathbb{S}}_T + 1)] \geq \mathcal{L}_N + B\right\}}.$$

The definitions of $h^{\tilde{\mathbb{Q}}}$, the set $\mathcal{M}^{\epsilon,\Lambda}_{\tilde{\kappa},\mathcal{L}}$ and the fact that $G \leq K$ implies that

$$\mathcal{G}(h^{\tilde{\mathbb{Q}}}, \tilde{\mathbb{Q}}) \leq 0, \qquad \forall\, \tilde{\mathbb{Q}} \in \mathcal{M}_{\tilde{\kappa}} \text{ but } \tilde{\mathbb{Q}} \notin \mathcal{M}^{\epsilon,\Lambda}_{\tilde{\kappa},\mathcal{L}}.$$

In particular, $\sup_{\tilde{\mathbb{Q}} \in \mathcal{M}_{\tilde{\kappa}}} \mathcal{G}(h^{\tilde{\mathbb{Q}}}, \tilde{\mathbb{Q}}) \leq 0$, if the set $\mathcal{M}^{\epsilon,\Lambda}_{\tilde{\kappa},\mathcal{L}}$ is empty. These together with (5.5) implies that

$$\begin{aligned} V_\kappa(G) &\leq \sup_{\tilde{\mathbb{Q}} \in \mathcal{M}_{\tilde{\kappa}}} \mathcal{G}(h^{\tilde{\mathbb{Q}}}, \tilde{\mathbb{Q}}) + L(e^{2\epsilon} + \epsilon - 1)\frac{\hat{C}^2}{2(1-8\kappa)} \\ &\leq \left(\sup_{\tilde{\mathbb{Q}} \in \mathcal{M}^{\epsilon,\Lambda}_{\tilde{\kappa},\mathcal{L}}} \mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{\mathbb{S}})]\right)^+ + L(e^{2\epsilon} + \epsilon - 1)\frac{\hat{C}^2}{2(1-8\kappa)}. \end{aligned}$$

□

6. Asymptotical Analysis of the Bounds

In this section. we complete the proof of Theorem 2.7. This is achieved by proving that the lower and the upper bounds from Sections 4 and 5 are asymptotically equal to each other.

Recall the probability measure $\mathbb{Q}$ from Assumption 2.3. Set, $D_i = \mathbb{E}_{\mathbb{Q}}[f_i(\mathbb{S})]$, $i \leq N$. Denote $\mathbf{D} = \prod_{i=1}^{N}(D_i, \infty)$. Let $H = (H_1, ..., H_N) \in \mathbf{D}$ and let $\tilde{\kappa} \in (0,1)$. Define $\mathcal{M}_{\tilde{\kappa},H}$ to be the set of all probability measures on $\hat{\Omega} := \Omega \times \mathcal{C}^{++}_{[0,T]}$ which satisfy the conditions of Definition 2.5, with $\kappa, \mathcal{L}_1, ..., \mathcal{L}_N$ replaced by $\tilde{\kappa}, H_1, ..., H_N$. Observe that $\mathbb{Q} \in \mathcal{M}_{\tilde{\kappa},H}$ and so, the set $\mathcal{M}_{\tilde{\kappa},H}$ is not empty. Define the function $\Gamma : \mathbf{D} \times (0,1) \to \mathbb{R}$ by

$$\Gamma(H, \tilde{\kappa}) := \sup_{\hat{\mathbb{Q}} \in \mathcal{M}_{\tilde{\kappa},H}} \mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})],$$

where, recall the canonical process $\hat{\mathbb{S}} = (\mathbb{S}^{(1)}_t, \mathbb{S}^{(2)}_t)_{0 \leq t \leq T}$ given in Definition 2.5. The following lemma is central in the analysis of the asymptotic behaviour of the bounds.

Lemma 6.1. *The function $\Gamma : \mathbf{D} \times (0,1) \to \mathbb{R}$ is continuous.*

Proof. It suffices to prove that for any compact set $J \subset \mathbf{D} \times (0,1)$ there exists a a continuous function $m_J : \mathbb{R}_+ \to \mathbb{R}_+$ (modulus of continuity) with $m_J(0) = 0$ such that for any $(H^{(i)}, \tilde{\kappa}_i) \in J$, $i = 1, 2$

$$\Gamma(H^{(1)}, \tilde{\kappa}_1) - \Gamma(H^{(2)}, \tilde{\kappa}_2) \leq m_J \left(\sum_{k=1}^{N} |H^{(1)}_k - H^{(2)}_k| + |\tilde{\kappa}_1 - \tilde{\kappa}_2| \right).$$

Choose $\epsilon > 0$. There exists $\hat{\mathbb{Q}}_1 \in \mathcal{M}_{\tilde{\kappa}_1, H^{(1)}}$ such that

$$\Gamma(H^{(1)}, \tilde{\kappa}_1) < \epsilon + \mathbb{E}_{\hat{\mathbb{Q}}_1}[G(\mathbb{S}^{(1)})]. \tag{6.1}$$

On the space $\hat{\Omega}$, define the stochastic processes ρ and $\dot{\rho}$ by,

$$\rho_t := \frac{\mathbb{S}^{(2)}_t}{\mathbb{S}^{(1)}_t} \quad \text{and} \quad \dot{\rho}_t := (1 - \tilde{\kappa}_2) \vee (\rho_t \wedge (1 + \tilde{\kappa}_2)), \quad t \in [0,T].$$

Next, introduce the stochastic process $\dot{\mathbb{S}} = (\dot{\mathbb{S}}^{(1)}_t, \dot{\mathbb{S}}^{(2)}_t)_{0 \leq t \leq T}$ by

$$\dot{\mathbb{S}}^{(1)}_t := \frac{\mathbb{S}^{(2)}_t}{\dot{\rho}_t} \frac{\dot{\rho}_0}{\rho_0} = \frac{\rho_t}{\dot{\rho}_t} \frac{\dot{\rho}_0}{\rho_0} \mathbb{S}^{(1)}_t \quad \text{and} \quad \dot{\mathbb{S}}^{(2)}_t := \frac{\dot{\rho}_0}{\rho_0} \mathbb{S}^{(2)}_t, \quad t \in [0,T].$$

Observe that there exists a constant $C^{(1)}_J$ such that

$$\sup_{0 \leq t \leq T} |\ln \dot{\mathbb{S}}^{(1)}_t - \ln \mathbb{S}^{(1)}_t| = \sup_{0 \leq t \leq T} |\ln \rho_t + \ln \dot{\rho}_0 - \ln \dot{\rho}_t - \ln \rho_0| \leq C^{(1)}_J |\tilde{\kappa}_1 - \tilde{\kappa}_2|. \tag{6.2}$$

Without loss of generality we assume that $C^{(1)}_J |\tilde{\kappa}_1 - \tilde{\kappa}_2| < \ln(1 + 1/L)$.

The idea behind the definition of the process $\dot{\mathbb{S}}$ is to construct a stochastic process which is "close" to $\mathbb{S}$ and satisfy properties (1) and (2) of Definition 2.5, for $\tilde{\kappa}_2$ instead of $\tilde{\kappa}_1$. In addition we require that $\dot{\mathbb{S}}^{(1)}_0 = 1$. Indeed, observe that $\dot{\mathbb{S}} : \hat{\Omega} \to \hat{\Omega}$. Thus, define the probability measure $\hat{\mathbb{Q}}_2$ to be the distribution of $\dot{\mathbb{S}}$ under the

probability measure $\hat{\mathbb{Q}}_1$. Namely, $\hat{\mathbb{Q}}_2$ is a probability measure on $\hat{\Omega}$ which is given by $\hat{\mathbb{Q}}_2(A) = \hat{\mathbb{Q}}_1(\dot{\mathbb{S}}^{-1}(A))$ for any Borel set $A \subset \hat{\Omega}$. Clearly, for any $t \in [0,T]$

$$(1-\tilde{\kappa}_2)\dot{\mathbb{S}}^{(1)}_t \leq \dot{\mathbb{S}}^{(2)}_t \leq (1+\tilde{\kappa}_2)\dot{\mathbb{S}}^{(1)}_t, \quad \hat{\mathbb{Q}}_1 \ \text{ a.s.}$$

and

$$\mathbb{E}_{\hat{\mathbb{Q}}_1}(\dot{\mathbb{S}}^{(2)}_T|\dot{\mathbb{S}}_u, \ u \leq t) = \dot{\mathbb{S}}^{(2)}_t.$$

Thus, for any $t \in [0,T]$,

$$(1-\tilde{\kappa}_2)\mathbb{S}^{(1)}_t \leq \mathbb{S}^{(2)}_t \leq (1+\tilde{\kappa}_2)\mathbb{S}^{(1)}_t, \quad \hat{\mathbb{Q}}_2 \ \text{ a.s.} \tag{6.3}$$

and

$$\mathbb{E}_{\hat{\mathbb{Q}}_2}(\mathbb{S}^{(2)}_T|\hat{\mathbb{F}}_t) = \mathbb{S}^{(2)}_t. \tag{6.4}$$

Next, similarly to (4.14) we obtain that there exists a constant $C^{(2)}_J$ such that

$$\mathbb{E}_{\hat{\mathbb{Q}}_1}[||\mathbb{S}^{(1)}||] \leq C^{(2)}_J.$$

By applying Assumptions 2.1–2.2 in a similar way to (4.12)–(4.13), and using (6.2) we obtain that we can construct another constant $C^{(3)}_J$ satisfying,

$$\begin{aligned}
|\mathbb{E}_{\hat{\mathbb{Q}}_2}[G(\mathbb{S}^{(1)})] - \mathbb{E}_{\hat{\mathbb{Q}}_1}[G(\mathbb{S}^{(1)})]| &= |\mathbb{E}_{\hat{\mathbb{Q}}_1}[G(\dot{\mathbb{S}}^{(1)})] - \mathbb{E}_{\hat{\mathbb{Q}}_1}[G(\mathbb{S}^{(1)})]| \\
&\leq LC^{(2)}_J(\exp(C^{(1)}_J|\tilde{\kappa}_1-\tilde{\kappa}_2|)-1) \\
&\leq C^{(3)}_J|\tilde{\kappa}_1-\tilde{\kappa}_2|
\end{aligned} \tag{6.5}$$

$$\begin{aligned}
|\mathbb{E}_{\hat{\mathbb{Q}}_2}[f_i(\mathbb{S}^{(1)})] - \mathbb{E}_{\hat{\mathbb{Q}}_1}[f_i(\mathbb{S}^{(1)})]| &= |\mathbb{E}_{\hat{\mathbb{Q}}_1}[f_i(\dot{\mathbb{S}}^{(1)})] - \mathbb{E}_{\hat{\mathbb{Q}}_1}[f_i(\mathbb{S}^{(1)})]| \\
&\leq LC^{(2)}_J(\exp(C^{(1)}_J|\tilde{\kappa}_1-\tilde{\kappa}_2|)-1) \\
&\leq C^{(3)}_J|\tilde{\kappa}_1-\tilde{\kappa}_2|, \quad i \leq N-1,
\end{aligned} \tag{6.6}$$

and for $i = N$

$$\begin{aligned}
|\mathbb{E}_{\hat{\mathbb{Q}}_2}[f_N(\mathbb{S}^{(1)})] &= \mathbb{E}_{\hat{\mathbb{Q}}_1}[f_N(\dot{\mathbb{S}}^{(1)})]| \\
&\leq \frac{\mathbb{E}_{\hat{\mathbb{Q}}_1}[f_N(\mathbb{S}^{(1)})](1+L(\exp(C^{(1)}_J|\tilde{\kappa}_1-\tilde{\kappa}_2|)-1))]}{1-L(\exp(C^{(1)}_J|\tilde{\kappa}_1-\tilde{\kappa}_2|)-1)} \\
&\quad + \frac{L(\exp(C^{(1)}_J|\tilde{\kappa}_1-\tilde{\kappa}_2|)-1)\mathbb{E}_{\hat{\mathbb{Q}}_1}[||\mathbb{S}^{(1)}||]}{1-L(\exp(C^{(1)}_J|\tilde{\kappa}_1-\tilde{\kappa}_2|)-1)} \\
&\leq \mathbb{E}_{\hat{\mathbb{Q}}_1}[f_N(\mathbb{S}^{(1)})] + C^{(3)}_J|\tilde{\kappa}_1-\tilde{\kappa}_2|.
\end{aligned} \tag{6.7}$$

Next, we modify the probability measure $\hat{\mathbb{Q}}_2$ so it will satisfy property (3) of Definition 2.5 for $H^{(2)}_1, ..., H^{(2)}_N$. Clearly, the measure $\mathbb{Q}\otimes\mathbb{Q}$ is a probability measure on $\hat{\Omega}$, where the probability measure $\mathbb{Q}$ is given in Assumption 2.3. For any $\lambda \in (0,1)$ consider the probability measure

$$\hat{\mathbb{Q}}_\lambda = \sqrt{\lambda}[\mathbb{Q}\otimes\mathbb{Q}] + (1-\sqrt{\lambda})\hat{\mathbb{Q}}_2.$$

Observe that

$$\mathbb{E}_{\mathbb{Q}\otimes\mathbb{Q}}[f_i(\mathbb{S}^{(1)})] = \mathbb{E}_{\mathbb{Q}}[f_i(\mathbb{S})] = D_i, \quad i \leq N.$$

Set $\Lambda=\sum_{k=1}^N|H^{(1)}_k-H^{(2)}_k|+|\tilde\kappa_1-\tilde\kappa_2|$. From (6.6)–(6.7) and the fact that $D_i<H^{(1)}_i$ it follows that for Λ sufficiently small

$$|\mathbb{E}_{\hat{\mathbb{Q}}_\Lambda}[f_i(\mathbb{S}^{(1)})]\le\sqrt{\Lambda}D_i+(1-\sqrt{\Lambda})(H^{(1)}_i+C^{(3)}_J|\tilde\kappa_1-\tilde\kappa_2|)\le$$
$$H^{(1)}_i-\sqrt{\Lambda}(H^{(1)}_i-D_i)+C^{(3)}_J\Lambda<H^{(1)}_i-\Lambda\le H^{(2)}_i.$$

This together with (6.3)–(6.4) yields that $\hat{\mathbb{Q}}_\Lambda\in\mathcal{M}_{\tilde\kappa_2,H^{(2)}}$. Finally, from (6.1) and (6.5) we obtain

$$\begin{aligned}\Gamma(H^{(1)},\tilde\kappa_1)-\Gamma(H^{(2)},\tilde\kappa_2)&\le\epsilon+\mathbb{E}_{\hat{\mathbb{Q}}_1}[G(\mathbb{S}^{(1)})]-(1-\sqrt{\Lambda})\mathbb{E}_{\hat{\mathbb{Q}}_2}[G(\mathbb{S}^{(1)})]\\&\le\epsilon+C^{(3)}_J|\tilde\kappa_1-\tilde\kappa_2|+\sqrt{\Lambda}K.\end{aligned}$$

Since $\epsilon>0$ was arbitrary, this completes the proof. □

Now, we are ready to prove the lower bound of Theorem 2.7.

Lemma 6.2.
$$V^{\mathbb{P}}_\kappa(G)\ge\sup_{\hat{\mathbb{Q}}\in\mathcal{M}_{\kappa,\mathcal{L}}}\mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})].$$

Proof. In view of Lemma 6.1, it is sufficient to prove that

$$V^{\mathbb{P}}_\kappa(G)\ge\mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})],\tag{6.8}$$

for every $\hat{\mathbb{Q}}\in\mathcal{M}_{\tilde\kappa,\tilde{\mathcal{L}}}$ with $\tilde\kappa<\kappa$ and $\tilde{\mathcal{L}}_i<\mathcal{L}_i$, $i\le N$.

We proceed in two steps. In the first step, we modify the process $\mathbb{S}^{(1)}$. In the second step, we apply Lemma 4.2 to the modified process.

First step: Let $\epsilon>0$. Define the stopping times, $\tau^{(\epsilon)}_0:=\tau^{(\epsilon)}_0(\mathbb{S}^{(1)})=0$ and for $k>0$,

$$\tau^{(\epsilon)}_k:=\tau^{(\epsilon)}_k(\mathbb{S}^{(1)})=T\wedge\inf\left\{t>\tau^{(\epsilon)}_{k-1}:\mathbb{S}^{(1)}_t=exp(\pm\epsilon)\mathbb{S}^{(1)}_{\tau^{(\epsilon)}_{k-1}}\right\},$$

and the random variable $\mathbb{K}:=\min\{k:\tau^{(\epsilon)}_k=T\}-1<\infty$. Let $n\in\mathbb{N}$. Introduce the stochastic process

$$\tilde S^{(n)}_t=\sum_{i=0}^{n-1}\mathbb{S}^{(1)}_{\tau^{(\epsilon)}_i}\chi_{[\tau^{(\epsilon)}_i,\tau^{(\epsilon)}_{i+1})}(t)+\mathbb{S}^{(1)}_{\tau^{(\epsilon)}_{\mathbb{K}\wedge n}}\chi_{[\tau^{(\epsilon)}_n,T]}(t),\quad t\in[0,T].$$

The stochastic process $\tilde S^{(n)}$ is a pure jump process which agrees with $\mathbb{S}^{(1)}$ at the jump times $\tau^{(\epsilon)}_1,...,\tau^{(\epsilon)}_{n\wedge\mathbb{K}}$ and remains constant afterwards.

We argue that for sufficiently large n the terms $\mathbb{E}_{\hat{\mathbb{Q}}}|f_i(\tilde S^{(n)})-f_i(\mathbb{S}^{(1)})|$, $i=1,...,N$ and $\mathbb{E}_{\hat{\mathbb{Q}}}|G(\tilde S^{(n)})-G(\mathbb{S}^{(1)})|$ are small. Indeed, as before the fact $\hat{\mathbb{Q}}\in\mathcal{M}_{\tilde\kappa,\tilde{\mathcal{L}}}$ implies that $\mathbb{E}_{\hat{\mathbb{Q}}}[\|\mathbb{S}^{(1)})\|]\le\hat C$ (where, recall the constant $\hat C$ from Definition 4.1) and so $\lim_{n\to\infty}\mathbb{E}_{\hat{\mathbb{Q}}}[\|\mathbb{S}^{(1)}\|\chi_{\{\mathbb{K}\ge n\}}]=0$. From Assumption 2.1 we get

$$\begin{aligned}\limsup_{n\to\infty}\left|\mathbb{E}_{\hat{\mathbb{Q}}}[f_i(\mathbb{S}^{(1)})]-\mathbb{E}_{\hat{\mathbb{Q}}}[f_i(\tilde S^{(n)})]\right|&\le\limsup_{n\to\infty}\mathbb{E}_{\hat{\mathbb{Q}}}[|f_i(\mathbb{S}^{(1)})-f_i(\tilde S^{(n)})|\chi_{\{\mathbb{K}<n\}}]\\&\quad+2L\lim_{n\to\infty}\mathbb{E}_{\hat{\mathbb{Q}}}[\|\mathbb{S}^{(1)}\|\chi_{\{\mathbb{K}\ge n\}}]\\&\le L(e^\epsilon-1)\mathbb{E}_{\hat{\mathbb{Q}}}[\|\mathbb{S}^{(1)})\|]\\&\le L(e^\epsilon-1)\hat C.\end{aligned}$$

Similarly,

$$\limsup_{n\to\infty}\left|\mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})]-\mathbb{E}_{\hat{\mathbb{Q}}}[G(\tilde{S}^{(n)})]\right|\leq L(e^{\epsilon}-1)\hat{C}. \tag{6.9}$$

It remains to treat the case $i=N$. From Assumption 2.2 it follows that there exists $\delta>0$ such that

$$|\ln x-\ln y|<\delta\Rightarrow q(y)<2(q(x)+x).$$

We conclude that there exists a constant C_4 such that or any $x,y>0$ we have

$$(1-\tilde{\kappa})x\leq y\leq\frac{1}{1-\tilde{\kappa}}x\Rightarrow q(y)\leq C_4(q(x)+x).$$

This together with property (2) of Definition 2.5 yields

$$\mathbb{E}_{\hat{\mathbb{Q}}}[q(\mathbb{S}^{(1)}_{\tau_n^{(\epsilon)}})\chi_{\{\mathbb{K}\geq n\}}]\leq C_4\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(q(\mathbb{S}^{(2)}_{\tau_n^{(\epsilon)}})+\mathbb{S}^{(2)}_{\tau_n^{(\epsilon)}}\right)\chi_{\{\mathbb{K}\geq n\}}\right].$$

Since $\mathbb{S}^{(2)}$ is a martingale and $\{\mathbb{K}\geq n\}=\{\tau_n^{(\epsilon)}<T\}\in\hat{\mathbb{F}}_{\tau_n^{(\epsilon)}}$, then from the Jensen inequality (for the convex function $q(x)+x$) we obtain,

$$\begin{aligned}\mathbb{E}_{\hat{\mathbb{Q}}}[q(\mathbb{S}^{(1)}_{\tau_n^{(\epsilon)}})\chi_{\{\mathbb{K}\geq n\}}]&\leq C_4\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(q(\mathbb{S}^{(2)}_T)+\mathbb{S}^{(2)}_T\right)\chi_{\{\mathbb{K}\geq n\}}\right]\\&\leq C_4\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(C_4q(\mathbb{S}^{(1)}_T)+(1+\tilde{\kappa})\mathbb{S}^{(1)}_T\right)\chi_{\{\mathbb{K}\geq n\}}\right].\end{aligned}$$

Thus the inequality $\mathbb{E}_{\hat{\mathbb{Q}}}[(q(\mathbb{S}^{(2)}_T)]<\infty$ implies

$$\begin{aligned}&\limsup_{n\to\infty}\left|\mathbb{E}_{\hat{\mathbb{Q}}}[f_N(\mathbb{S}^{(1)})]-\mathbb{E}_{\hat{\mathbb{Q}}}[f_N(\tilde{S}^{(n)})]\right|\\&\qquad\leq\limsup_{n\to\infty}\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(f_N(\mathbb{S}^{(1)})+f_N(\tilde{S}^{(n)})\right)\chi_{\{\mathbb{K}\geq n\}}\right]\\&\qquad\leq\limsup_{n\to\infty}C_4\mathbb{E}_{\hat{\mathbb{Q}}}\left[(C_4+1/C_4)\left(q(\mathbb{S}^{(1)}_T)+(1+\tilde{\kappa})\mathbb{S}^{(1)}_T\right)\chi_{\{\mathbb{K}\geq n\}}\right]\\&\qquad=0.\end{aligned}$$

We conclude that for sufficiently large n

$$\begin{aligned}&\left|\mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})]-\mathbb{E}_{\hat{\mathbb{Q}}}[G(\tilde{S}^{(n)})]\right|\leq 2L(e^{\epsilon}-1)\hat{C}\ \text{ and}\\&\left|\mathbb{E}_{\hat{\mathbb{Q}}}[f_i(\mathbb{S}^{(1)})]-\mathbb{E}_{\hat{\mathbb{Q}}}[f_i(\tilde{S}^{(n)})]\right|\leq 2L(e^{\epsilon}-1)\hat{C}\ \ i\leq N.\end{aligned} \tag{6.10}$$

We fix n sufficiently large that the above inequalities hold and set $\tilde{S}:=\tilde{S}^{(n)}$.

Next, we modify the jump times so they will lie on a grid. Let $m\in\mathbb{N}$. Define by recursion the following sequence of random variables,

$$\begin{aligned}\hat{\tau}_k^{(\epsilon)}&:=\sum_{i=1}^{k}\Delta\hat{\tau}_i^{(\epsilon)},\quad\text{where}\\\Delta\hat{\tau}_i^{(\epsilon)}&=\min\{\Delta t\in\{T/m,2T/m,...,T\}:\Delta t\geq\Delta\tau_i^{(\epsilon)}:=\tau_i^{(\epsilon)}-\tau_{i-1}^{(\epsilon)}\},\end{aligned}$$

and

$$\sigma_k=T\chi_{\{\tau_k^{(\epsilon)}=T\}}+\hat{\tau}_k^{(\epsilon)}\wedge(T(1-2^{-k}/m))\chi_{\{\tau_k^{(\epsilon)}<T\}},\quad k=0,1,...,n.$$

Observe that for any i, $\sigma_{i+1} \geq \sigma_i$ and $\sigma_{i+1} = \sigma_i$ if and only if $\sigma_i = T$. Notice that $\sigma_1, ..., \sigma_n$ are not (in general) stopping times with respect to the filtration $\hat{\mathbb{F}}$. Define the stochastic process

$$\dot{S}_t := \dot{S}^{(m)}_t = \sum_{i=0}^{n-1} \mathbb{S}^{(1)}_{\tau^{(\epsilon)}_i} \chi_{[\sigma_i, \sigma_{i+1})}(t) + \mathbb{S}^{(1)}_{\tau^{(\epsilon)}_{\mathbb{K}\wedge n}} \chi_{[\sigma_n, T]}(t), \quad t \in [0, T].$$

Second step: The process $\dot{S}_t$ is a piecewise constant process, and the jump times are lying on a finite grid. Thus the natural filtration which is generated by $\dot{S}$ is right continuous, and so the martingale

$$\hat{M}_t := \mathbb{E}_{\hat{\mathbb{Q}}}(\mathbb{S}^{(2)}_T | \dot{S}_u, u \leq t)$$

is a *càdlàg* martingale. Let $k \leq n$. Clearly, σ_k is a stopping time with respect to the natural filtration generated by $\dot{S}$. Furthermore $\dot{S}_{[0,\sigma_k]}$ is measurable with respect to $\hat{\mathbb{F}}_{\tau^{(\epsilon)}_k}$. This together with the fact that

$$e^{-\epsilon} \leq \frac{\dot{S}_{\sigma_k}}{\mathbb{S}^{(1)}_{\tau^{(\epsilon)}_k}} \leq e^{\epsilon}$$

and properties (1)–(2) in Definition 2.5, imply that

$$\begin{aligned} |\hat{M}_{\sigma_k} - \dot{S}_{\sigma_k}| &= \left| \mathbb{E}_{\hat{\mathbb{Q}}} \left(\mathbb{E}_{\hat{\mathbb{Q}}}[\mathbb{S}^{(2)}_T \mid \hat{\mathbb{F}}_{\tau^{(\epsilon)}_k}] \mid \dot{S}_u, u \leq \sigma_k \right) - \dot{S}_{\sigma_k} \right| \\ &\leq \dot{S}_{\sigma_k}((1+\tilde{\kappa})e^{\epsilon} - 1) \leq \dot{S}_{\sigma_k}(\tilde{\kappa} + 2\epsilon), \end{aligned}$$

where in the last equality we assume that ϵ is sufficiently small. Let $\sigma_{n+1} = T$. Then, for any $k \leq n$ and $t \in [\sigma_k, \sigma_{k+1}]$, we conclude that

$$e^{-2\epsilon}(1 - \tilde{\kappa} - 2\epsilon)\dot{S}_t \leq \hat{M}_{\sigma_{k+1}} \leq e^{2\epsilon}(1 + \tilde{\kappa} + 2\epsilon)\dot{S}_t.$$

Since $\hat{M}$ is a martingale with respect to the natural filtration of $\dot{S}$, we conclude that for sufficiently small ϵ,

$$|\hat{M}_t - \dot{S}_t| \leq (1 + \tilde{\kappa} + 5\epsilon)\dot{S}_t. \tag{6.11}$$

Clearly,

$$\lim_{m\to\infty} \|\tilde{S} - \dot{S}^{(m)}\| = 0, \quad \hat{\mathbb{Q}} \text{ a.s.}$$

Observe that the above processes are uniformly bounded. Hence, by Assumptions 2.1–2.2,

$$\begin{aligned} \mathbb{E}_{\hat{\mathbb{Q}}}[G(\tilde{S})] &= \lim_{m\to\infty} \mathbb{E}_{\hat{\mathbb{Q}}}[G(\dot{S}^{(m)})] \text{ and} \\ \mathbb{E}_{\hat{\mathbb{Q}}}[f_i(\tilde{S})] &= \lim_{m\to\infty} \mathbb{E}_{\hat{\mathbb{Q}}}[f_i(\dot{S}^{(m)})], \quad i \leq N. \end{aligned} \tag{6.12}$$

Denote by $\dot{\mathbb{Q}}_m$ the distribution of $\dot{S}^{(m)}$ on the space $\mathbb{D}[0, T]$. Let us choose ϵ such that $\hat{\kappa} := \tilde{\kappa} + 6\epsilon$ is satisfies

$$\min\left(\frac{1+\kappa}{1+\hat{\kappa}}, \frac{1-\hat{\kappa}}{1-\kappa}\right) \geq e^{2\epsilon},$$

and

$$\mathcal{L}_i - L(\hat{C} + \mathcal{L}_N)(e^{4\epsilon} + \epsilon - 1) > 3L(e^{\epsilon} - 1)\hat{C} + \tilde{\mathcal{L}}_i, \quad i < N,$$

$$\frac{\mathcal{L}_N(1 - L(e^{\epsilon} - 1)) - L\hat{C}(e^{\epsilon} - 1)}{1 + L(e^{\epsilon} - 1)} > 3L(e^{\epsilon} - 1)\hat{C} + \tilde{\mathcal{L}}_N.$$

From (6.10)–(6.12), it follows that for sufficiently large m the measure $\dot{\mathbb{Q}}_m \in \mathcal{M}^{\mathcal{T},\epsilon}_{\hat{\kappa},\mathcal{L}}$ with the choice $\mathcal{T} := \{kT2^{-n}/m\}_{k=0}^{2^n m}$. Thus, in view of Lemma 4.2, we have

$$V^{\mathbb{P}}_\kappa(G) \geq \mathbb{E}_{\hat{\mathbb{Q}}}[G(\dot{S}^{(m)})] - L\hat{C}(e^{4\epsilon} + \epsilon - 1).$$

We now apply (6.10), (6.12) and take the limit as m tends to infinity. The result is

$$V^{\mathbb{P}}_\kappa(G) \geq \mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})] - 2L(e^{\epsilon} - 1)\hat{C} - L\hat{C}(e^{4\epsilon} + \epsilon - 1).$$

Now, (6.8) follows after taking the limit as ϵ tends to zero. □

Next, we establish the upper bound (2.4).

Lemma 6.3.

$$V_\kappa(G) \leq \sup_{\hat{\mathbb{Q}} \in \mathcal{M}_{\kappa,\mathcal{L}}} \mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})].$$

Proof. Let $\mathbb{Q}$ be the probability measure from Assumption 2.3. Then, $\mathbb{Q} \otimes \mathbb{Q} \in \mathcal{M}_{\kappa,(\mathcal{L}_1,\ldots,\mathcal{L}_N)}$. Therefore, if $V_\kappa(G) \leq 0$, then (2.4) is trivial. So we may assume without loss of generality that $V_\kappa(G) > 0$. Choose $\epsilon > 0$, $\Lambda > 1$, $\hat{\kappa} > \tilde{\kappa} > \kappa$ and $\tilde{\mathcal{L}}_i > \mathcal{L}_i$, $i \leq N$. Assume that ϵ is sufficiently small so $L(e^{2\epsilon} + \epsilon - 1)\frac{\hat{C}^2}{2(1-8\kappa)} < V_\kappa(G)$ and $\tilde{\kappa}$ satisfies (5.1). This together with Lemma 5.3 yields that there exists a probability measure $\tilde{\mathbb{Q}} \in \mathcal{M}^{\epsilon,\Lambda}_{\tilde{\kappa},\mathcal{L}}$ such that

$$V_\kappa(G) < \mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{\mathbb{S}})] + L(e^{2\epsilon} + \epsilon - 1)\frac{\hat{C}^2}{(1-8\kappa)}. \tag{6.13}$$

Next, we proceed in three steps. In the first step (similarly to Lemma 6.2), we modify the stochastic process $\tilde{\mathbb{S}}$. In the second step, we use the Wiener space in order to construct a continuous consistent price system with (almost) the required properties. In the last step, we modify again the constructed continuous consistent price system in order to get rid of the truncation in the term $f_N(\mathbb{S}^{(1)}) \wedge \Lambda \mathbb{S}^{(1)}_T$. Finally, we Apply Lemma 6.1.

First step: Let

$$(1-\tilde{\kappa})\tilde{\mathbb{S}}_t \leq \tilde{M}_t \leq (1+\tilde{\kappa})\tilde{\mathbb{S}}_t, \quad t \in [0,T],$$

be the associated martingale corresponding to the probability measure $\tilde{\mathbb{Q}} \in \mathcal{M}^{\epsilon,\Lambda}_{\tilde{\kappa},\mathcal{L}}$. Let $\tilde{\tau}^{(\epsilon)}_0 := \tilde{\tau}^{(\epsilon)}_0(\tilde{\mathbb{S}}) = 0$, and for $k > 0$ set,

$$\tilde{\tau}^{(\epsilon)}_k := \tilde{\tau}^{(\epsilon)}_k(\tilde{\mathbb{S}}) = T \wedge \inf\left\{t > \tilde{\tau}^{(\epsilon)}_{k-1} : |\ln \tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_{k+1}} - \ln \tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_k}| = \epsilon\right\}$$

and $\tilde{\mathbb{K}} = \min\{k : \tilde{\tau}^{(\epsilon)}_k = T\} - 1 < \infty$. Observe that the probability measure $\tilde{\mathbb{Q}}$ supported on $\mathbb{D}^{(\epsilon)}$ and so $\tilde{\tau}_k$, $k \geq 0$ are indeed stopping times.

Let $n \in \mathbb{N}$. Set,

$$\tilde{S}^{(n)}_t := \sum_{i=0}^{n-1} \tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_i} \chi_{[\tilde{\tau}^{(\epsilon)}_i, \tilde{\tau}^{(\epsilon)}_{i+1})}(t) + \tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_{\tilde{\mathbb{K}}\wedge n}} \chi_{[\tilde{\tau}^{(\epsilon)}_n, T]}(t), \quad t \in [0,T].$$

From the definition of the set $\mathcal{M}^{\epsilon,\Lambda}_{\tilde{\kappa},\mathcal{L}}$ it follows that $\mathbb{E}_{\tilde{\mathbb{Q}}}[q(\tilde{\mathbb{S}}_T) \wedge \Lambda(\tilde{\mathbb{S}}_T + 1)] < \infty$, and so and $\mathbb{E}_{\tilde{\mathbb{Q}}}[\tilde{\mathbb{S}}_T] < \infty$, as well. Moreover,

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{Q}}}[\tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_n} \chi_{\{\tilde{\mathbb{K}} \geq n\}}] &\leq (1+\tilde{\kappa})\mathbb{E}_{\tilde{\mathbb{Q}}}[\tilde{M}_{\tilde{\tau}^{(\epsilon)}_n} \chi_{\{\tilde{\mathbb{K}} \geq n\}}] = (1+\tilde{\kappa})\mathbb{E}_{\tilde{\mathbb{Q}}}[\tilde{M}_T \chi_{\{\tilde{\mathbb{K}} \geq n\}}] \\ &\leq (1+\tilde{\kappa})^2 \mathbb{E}_{\tilde{\mathbb{Q}}}[\tilde{\mathbb{S}}_T \chi_{\{\tilde{\mathbb{K}} \geq n\}}]. \end{aligned}$$

We conclude that

$$\lim_{n\to\infty} \mathbb{E}_{\tilde{\mathbb{Q}}}[(\tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_n} + \tilde{\mathbb{S}}_T)\chi_{\{\tilde{\mathbb{K}}\geq n\}}] = 0. \tag{6.14}$$

As in the proof of Lemma 5.3, we will use the fact that $f_i(S)$, $i < N$ are bounded (from both sides) by a multiply of $1 + S_T$. This together with (6.14) and the fact that $\tilde{S}^{(n)} = \tilde{\mathbb{S}}$ on the event $\{n > \tilde{\mathbb{K}}\}$ yields that for sufficiently large n,

$$\begin{aligned}
&\left|\mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{\mathbb{S}})] - \mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{S}^{(n)})]\right| \leq \epsilon,\\
&\left|\mathbb{E}_{\tilde{\mathbb{Q}}}[f_i(\tilde{\mathbb{S}})] - \mathbb{E}_{\tilde{\mathbb{Q}}}[f_i(\tilde{S}^{(n)})]\right| \leq \epsilon, \quad i \leq N-1,\\
&\left|\mathbb{E}_{\tilde{\mathbb{Q}}}[q(\tilde{\mathbb{S}}_T) \wedge \Lambda(\tilde{\mathbb{S}}_T + 1)] - \mathbb{E}_{\tilde{\mathbb{Q}}}[q(\tilde{S}^{(n)}_T) \wedge \Lambda(\tilde{S}^{(n)}_T + 1)]\right| \leq \epsilon.
\end{aligned} \tag{6.15}$$

We choose n sufficiently large and set $\tilde{S} := \tilde{S}^{(n)}$.

Next, let $m \in \mathbb{N}$. Define by recursion the following sequence of random variables,

$$\begin{aligned}
\hat{\tau}^{(\epsilon)}_k &:= \sum_{i=1}^{k} \Delta\hat{\tau}^{(\epsilon)}_i, \quad \text{where}\\
\Delta\hat{\tau}^{(\epsilon)}_i &= \min\{\Delta t \in \{T/m, 2T/m, ..., T\} : \Delta t \geq \Delta\tilde{\tau}^{(\epsilon)}_i := \tilde{\tau}^{(\epsilon)}_i - \tilde{\tau}^{(\epsilon)}_{i-1}\},
\end{aligned}$$

and

$$\sigma_k = T\chi_{\{\tilde{\tau}^{(\epsilon)}_k = T\}} + \hat{\tau}^{(\epsilon)}_k \wedge (T(1-2^{-k}/m))\chi_{\{\tilde{\tau}^{(\epsilon)}_k < T\}}, \quad k = 0, 1, ..., n.$$

Similarly, to Lemma 6.2 we have that for any i, $\sigma_{i+1} \geq \sigma_i$ and $\sigma_{i+1} = \sigma_i$ if and only if $\sigma_i = T$. Define the stochastic process

$$\dot{S}_t := \dot{S}^{(m)}_t = \sum_{i=0}^{n-1} \tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_i}\chi_{[\sigma_i,\sigma_{i+1})}(t) + \tilde{\mathbb{S}}_{\tilde{\tau}^{(\epsilon)}_{\tilde{\mathbb{K}}\wedge n}}\chi_{[\sigma_n,T]}(t), \quad t \in [0,T].$$

Again, as in Lemma 6.2 the process $\dot{S}_t$ is a piecewise constant process, and the jump times are lying on a finite grid. Introduce the (*càdlàg*) martingale

$$\hat{M}_t := \mathbb{E}_{\hat{\mathbb{Q}}}(\tilde{M}_T | \dot{S}_u, u \leq t).$$

By using the same arguments as in (6.11)–(6.12) we get

$$|\hat{M}_t - \dot{S}_t| \leq (1 + \tilde{\kappa} + 5\epsilon)\dot{S}_t, \tag{6.16}$$

and

$$\begin{aligned}
\mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{S})] &= \lim_{m\to\infty} \mathbb{E}_{\tilde{\mathbb{Q}}}[G(\dot{S}^{(m)})],\\
\mathbb{E}_{\tilde{\mathbb{Q}}}[f_i(\tilde{S})] &= \lim_{m\to\infty} \mathbb{E}_{\tilde{\mathbb{Q}}}[f_i(\dot{S}^{(m)})], \quad i \leq N-1\\
\mathbb{E}_{\tilde{\mathbb{Q}}}[q(\tilde{S}_T) \wedge \Lambda(\tilde{\mathbb{S}}_T + 1)] &= \lim_{m\to\infty} \mathbb{E}_{\tilde{\mathbb{Q}}}[q(\dot{S}^{(m)}_T) \wedge \Lambda(\dot{S}^{(m)}_T + 1)].
\end{aligned} \tag{6.17}$$

From (6.15) and (6.17), it follows that we can choose m sufficiently large such that

$$\begin{aligned}
&\left|\mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{\mathbb{S}})] - \mathbb{E}_{\tilde{\mathbb{Q}}}[G(\dot{S}^{(m)})]\right| \leq 2\epsilon,\\
&\left|\mathbb{E}_{\tilde{\mathbb{Q}}}[f_i(\tilde{\mathbb{S}})] - \mathbb{E}_{\tilde{\mathbb{Q}}}[f_i(\dot{S}^{(m)})]\right| \leq 2\epsilon, \quad i \leq N-1,\\
&\left|\mathbb{E}_{\tilde{\mathbb{Q}}}[q(\tilde{\mathbb{S}}_T) \wedge \Lambda(\tilde{\mathbb{S}}_T + 1)] - \mathbb{E}_{\tilde{\mathbb{Q}}}[q(\dot{S}^{(m)}_T) \wedge \Lambda(\dot{S}^{(m)}_T + 1)]\right| \leq 2\epsilon.
\end{aligned} \tag{6.18}$$

Choose such m and denote $\dot{S} = \dot{S}^{(m)}$. The stochastic process $\{\dot{S}_t\}_{t=0}^{T}$ is a piecewise constant process, and the jump times are lying on a finite grid. Denote the grid by $\mathcal{T} = \{t_1, ..., t_r, T\}$, where $0 = t_0 < t_1 < ... < t_r < T$.

Second step: Let $(\Omega^W, \mathcal{F}^W, \mathbb{P}^W)$ be a complete probability space together with a standard Brownian motion and the natural filtration $\mathcal{F}_t^W = \sigma\{W_s | s \le t\}$.

From Theorem 1 in Skorokhod (1976) and the fact that the random variables $W_{t_{i+1}} - W_{t_i}$, $i = 0, .., r-1$ are independent, it follows that we can find a sequence of measurable function $g_i^{(1)}, g_i^{(2)} : \mathbb{R}^{2i-1} \to \mathbb{R}$, $i = 1, ..., r$ with the following property. The stochastic processes (adapted to the Brownian filtration) $\{\dot{S}_{t_i}^W\}_{i=0}^{r}$ and $\{\hat{M}_{t_i}^W\}_{i=0}^{r}$ which are given by the recursion relations

$$\dot{S}_{t_0}^W = 1, \quad \hat{M}_{t_0}^W = \hat{M}_0$$

and for $i > 0$

$$\begin{aligned}
\dot{S}_{t_i}^W &= g_i^{(1)}(W_{t_{i+1}} - W_{t_i}, \dot{S}_{t_0}^W, ..., \dot{S}_{t_{i-1}}^W, \hat{M}_{t_0}^W, ..., \hat{M}_{t_{i-1}}^W), \\
\hat{M}_{t_i}^W &= g_i^{(2)}(W_{t_{i+1}} - W_{t_i}, \dot{S}_{t_0}^W, ..., \dot{S}_{t_{i-1}}^W, \hat{M}_{t_0}^W, ..., \hat{M}_{t_{i-1}}^W)
\end{aligned}$$

have the same joint distribution as the processes $\{\dot{S}_{t_i}\}_{i=0}^{r}$ and $\{\hat{M}_{t_i}\}_{i=0}^{r}$. Namely, the distribution of

$$(\dot{S}_{t_0}^W, ..., \dot{S}_{t_r}^W, \hat{M}_{t_0}^W, ..., \hat{M}_{t_r}^W)$$

under the probability measure $\mathbb{P}^W$ is equals to the distribution of

$$(\dot{S}_{t_0}, ..., \dot{S}_{t_r}, \hat{M}_{t_0}, ..., \hat{M}_{t_r})$$

under the probability measure $\tilde{\mathbb{Q}}$.

Since the Brownian motion increments are independent, for any $i < r$,

$$\mathbb{E}_{\mathbb{P}^W}(\hat{M}_{t_{i+1}}^W | \mathcal{F}_{t_i}^W) = \mathbb{E}_{\mathbb{P}^W}(\hat{M}_{t_{i+1}}^W | \dot{S}_{t_1}^W, ..., \dot{S}_{t_i}^W, \hat{M}_{t_1}^W, ..., \hat{M}_{t_i}^W) = \hat{M}_{t_i}^W.$$

Thus, we can extend the martingale $\{\hat{M}_{t_i}^W\}_{i=0}^{r}$ to a continuous time martingale (Brownian martingale)

$$\hat{M}_t^W = \mathbb{E}_{\mathbb{P}^W}(\hat{M}_{t_r}^W | \mathcal{F}_t^W), \quad t \in [0, T].$$

Next, we define the stochastic process $\{S_t^W\}_{t=0}^{T}$ by the following linear interpolation,

$$S_t^W = \chi_{[0,t_1]}(t) + \sum_{i=1}^{r} \frac{(t - t_i)\dot{S}_{t_i}^W + (t_{i+1} - t)\dot{S}_{t_{i-1}}^W}{t_{i+1} - t_i} \chi_{(t_i, t_{i+1}]}(t),$$

where we set $t_{r+1} = T$. Observe that the stochastic process S^W is continuous and adapted to the Brownian filtration. Since

$$\frac{\dot{S}_{t_{i+1}}^W}{\dot{S}_{t_i}^W} \in \{1, e^{\epsilon}, e^{-\epsilon}\},$$

it follows from (6.16) that (for ϵ sufficiently small)

$$\left| \hat{M}_t^W - S_t^W \right| \le (\tilde{\kappa} + 10\epsilon) S_t^W, \quad t \in [0, T]. \tag{6.19}$$

Set,

$$\dot{S}_t^W = \sum_{i=0}^{r-1} \dot{S}_{t_i}^W \chi_{[t_i, t_{i+1})}(t) + \dot{S}_{t_r}^W \chi_{[t_r, T]}(t), \quad t \in [0, T].$$

Clearly, the processes $\dot{S}^W$ and $\dot{S}$ have the same distribution and and consequently,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}^W}[G(\dot{S}^W)] &= \mathbb{E}_{\tilde{\mathbb{Q}}}[G(\dot{S})], \\
\mathbb{E}_{\mathbb{P}^W}[f_i(\dot{S}^W)] &= \mathbb{E}_{\tilde{\mathbb{Q}}}[f_i(\dot{S})], \quad i \le N-1, \\
\mathbb{E}_{\mathbb{P}^W}[q(\dot{S}_T^W) \wedge \Lambda(\dot{S}_T^W + 1)] &= \mathbb{E}_{\tilde{\mathbb{Q}}}[q(\dot{S}_T) \wedge \Lambda(\dot{S}_T + 1)].
\end{aligned} \tag{6.20}
$$

Also, (6.18) and (6.20) imply that

$$\mathbb{E}_{\mathbb{P}^W}[q(\dot{S}_T^W) \wedge \Lambda(\dot{S}_T^W + 1)] \le 2\epsilon + \mathcal{L}_N + B,$$

where B is given in Definition 5.2. Therefore, there exists a constant C (which does not depend on $\epsilon > 0$ and $\Lambda > 1$) such that $\mathbb{E}_{\mathbb{P}^W}\dot{S}_T^W \le C$. This together with the Kolmogorov inequality for the martingale $\hat{M}^W$ yield that

$$
\begin{gathered}
\mathbb{P}^W\left(\|S^W\| > \tfrac{1}{\sqrt{\epsilon}}\right) \le \mathbb{P}^W\left(\|\hat{M}^W\| > \tfrac{1}{(1+\tilde{\kappa}+10\epsilon)\sqrt{\epsilon}}\right) \le \\
\mathbb{E}_{\mathbb{P}^W}[\hat{M}_T^W(1+\tilde{\kappa}+10\epsilon)\sqrt{\epsilon}] \le C(1+\tilde{\kappa}+10\epsilon)^2\sqrt{\epsilon}.
\end{gathered}
$$

Observe that by construction $\|S^W - \dot{S}^W\| \le 4\epsilon||S^W||$. Thus from Assumption 2.1 it follows that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}^W}[|G(S^W) - G(\dot{S}^W)|] &\le \mathbb{E}_{\mathbb{P}^W}[K\chi_{\{|S^W\|>1/\sqrt{\epsilon}\}} + 4L\sqrt{\epsilon}\chi_{\{|S^W\|\le 1/\sqrt{\epsilon}\}}] \\
&\le (KC(1+\tilde{\kappa}+10\epsilon)^2 + 4L)\sqrt{\epsilon}.
\end{aligned}
$$

Similarly for path dependent f_i we have

$$\mathbb{E}_{\mathbb{P}^W}[|f_i(S^W) - f_i(\dot{S}^W)|] \le (2\|f_i\|_\infty C(1+\tilde{\kappa}+10\epsilon)^2 + 4L)\sqrt{\epsilon}$$

where $\|f_i\|_\infty$ is the uniform bound of the path dependent claim $|f_i|$. Since $S_T^W = \dot{S}_T^W$ then for non path dependent f_i we have a trivial estimate. We now use these inequalities together with (6.18) and (6.20), to construct a constant $\tilde{C}$ satisfying,

$$
\begin{aligned}
&\left|\mathbb{E}_{\mathbb{P}^W}[G(S^W)] - \mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{\mathbb{S}})]\right| \le \tilde{C}\sqrt{\epsilon}, \\
&\left|\mathbb{E}_{\mathbb{P}^W}[f_i(S^W)] - \mathbb{E}_{\tilde{\mathbb{Q}}}[f_i(\tilde{\mathbb{S}})]\right| \le \tilde{C}\sqrt{\epsilon}, \quad i \le N-1, \\
&\left|\mathbb{E}_{\tilde{\mathbb{Q}}}\left[q(S_T^W) \wedge \Lambda(S_T^W+1)\right] - \mathbb{E}_{\tilde{\mathbb{Q}}}\left[q(\tilde{\mathbb{S}}_T) \wedge \Lambda(\tilde{\mathbb{S}}_T + 1)\right]\right| \le \tilde{C}\sqrt{\epsilon}.
\end{aligned} \tag{6.21}
$$

Third step: Let x_Λ be the solution of the equation $q(x) = \Lambda(x+1)$ where we assume that $\Lambda > q(0)$ so the equation has exactly one solution. Indeed (if by contradiction) we have two solutions $0 < x < y$ then

$$\frac{q(y) - q(x)}{y - x} = \Lambda < \frac{q(x) - q(0)}{x}$$

and we get contradiction to convexity. Define the stochastic processes by,

$$\rho_t := \frac{\hat{M}_t^W}{S_t^W}, \qquad M_t := \mathbb{E}_{\mathbb{P}^W}(\hat{M}_T^W \wedge \rho_T x_\Lambda | \mathcal{F}_t^W),$$

and

$$S_t := \frac{M_t}{\rho_t}\frac{t + (T-t)\rho_0/M_0}{T}, \quad t \in [0, T].$$

In view of (6.21),

$$\mathbb{E}_{\mathbb{P}^W}\left[\hat{M}^W_T \chi_{\hat{M}^W_T>\rho_T x_\Lambda}\right] \leq 2\mathbb{E}_{\mathbb{P}^W}\left[S^W_T \chi_{S^W_T>x_\Lambda}\right] \leq \frac{2}{\Lambda}\mathbb{E}_{\tilde{\mathbb{Q}}}\left[q(S^W_T)\wedge\Lambda(S^W_T+1)\right] \leq \frac{2(\tilde{C}\sqrt{\epsilon}+\mathcal{L}_N+B)}{\Lambda}. \tag{6.22}$$

Thus $|M_0-\rho_0| = |M_0-\hat{M}^W_0|\leq C_1/\Lambda$ for some constant C_1. This together with (6.19) implies that for sufficiently large Λ we have the following inequality,

$$|M_t - S_t| \leq \left(\tilde{\kappa}+10\epsilon+\frac{1}{\sqrt{\Lambda}}\right)S_t, \quad t\in[0,T]. \tag{6.23}$$

Next, consider the martingale

$$m_t := \mathbb{E}_{\mathbb{P}^W}\left[\hat{M}^W_T \chi_{\{\hat{M}^W_T>\rho_T x_\Lambda\}}\,|\mathcal{F}^W_t\right], \quad t\in[0,T].$$

Observe that $0\leq \hat{M}^W_t - M_t \leq m_t$, $t\in[0,T]$. Thus we obtain that there exists a constant C_2 such that

$$\|S^W - S\| \leq |M^W - M\| \sup_{0\leq t\leq T}\frac{1}{\rho_t} + \|M\|\sup_{0\leq t\leq T}\left|\frac{1}{\rho_t} - \frac{t+(T-t)\rho_0/M_0}{T\rho_t}\right| \leq 2\|m\| + \frac{C_2}{\Lambda}\|M^W\|. \tag{6.24}$$

The Kolmogorov inequality and (6.22) imply that

$$\mathbb{P}^W\left(\|m\|>1/\sqrt{\Lambda}\right)\leq C_3/\sqrt{\Lambda}$$

for some constant C_3. Moreover,

$$\mathbb{P}^W\left(\|M^W\|>\sqrt{\Lambda}\right)\leq \frac{M^W_0}{\sqrt{\Lambda}}\leq\frac{2}{\sqrt{\Lambda}}.$$

From (6.24) we conclude that

$$\mathbb{P}^W\left(\|S^W-S\|>\frac{2+C_2}{\sqrt{\Lambda}}\right)\leq\frac{2+C_3}{\sqrt{\Lambda}}.$$

Thus from Assumption 2.2 it follows that

$$\mathbb{E}_{\mathbb{P}^W}[|G(S^W)-G(S)|] \leq \mathbb{E}_{\mathbb{P}^W}[K\chi_{\{|S^W-S\|>\frac{2+C_2}{\sqrt{\Lambda}}\}} + L\tfrac{2+C_2}{\sqrt{\Lambda}}\chi_{\{|S^W\|\leq\frac{2+C_2}{\sqrt{\Lambda}}\}}] \leq \tfrac{C_4}{\sqrt{\Lambda}} \tag{6.25}$$

for some constant C_4. Similarly for path–dependent f_i we get

$$\mathbb{E}_{\mathbb{P}^W}[|f_i(S^W)-f_i(S)|]\leq\frac{C_4}{\sqrt{\Lambda}}. \tag{6.26}$$

For non path–dependent f_i, $i<N$ we have

$$\mathbb{E}_{\mathbb{P}^W}[|f_i(S^W)-f_i(S)|] \leq L\mathbb{E}_{\mathbb{P}^W}[|S^W_T - S_T|] \leq L\mathbb{E}_{\mathbb{P}^W}\left[S^W_T\chi_{S^W_T>x_\Lambda}\right] \leq \frac{L(\tilde{C}\sqrt{\epsilon}+\mathcal{L}_N+B)}{\Lambda} \tag{6.27}$$

where the last inequality follows from (6.22). The only remanning delicate point is $i = N$. From the fact that $S_T = S^W_T \wedge x_\Lambda$ we get

$$\mathbb{E}_{\mathbb{P}^W}[q(S_T)] \le \mathbb{E}_{\mathbb{P}^W}[q(S^W_T) \wedge \Lambda(S^W_T + 1)].$$

This together with (6.21), (6.23) and (6.25)–(6.27) yields that for sufficiently large Λ and small $\epsilon > 0$ the distribution of (S, M) on the space $\hat{\Omega} := \Omega \times \mathcal{C}^{++}_{[0,T]}$ is an element in $\mathcal{M}_{\hat{\kappa},(\tilde{\mathcal{L}}_1,\ldots,\tilde{\mathcal{L}}_N)}$. Furthermore,

$$\Big|\mathbb{E}_{\mathbb{P}^W}[G(S)] - \mathbb{E}_{\tilde{\mathbb{Q}}}[G(\tilde{\mathbb{S}})]\Big| \le \tilde{C}\sqrt{\epsilon} + \frac{C_4}{\sqrt{\Lambda}}.$$

We now use (6.13), to obtain

$$V_\kappa(G) < L(e^{2\epsilon} + \epsilon - 1)\frac{\hat{C}^2}{(1-8\kappa)} + \tilde{C}\sqrt{\epsilon} + \frac{C_4}{\sqrt{\Lambda}} + \sup_{\hat{\mathbb{Q}} \in \mathcal{M}_{\hat{\kappa},(\tilde{\mathcal{L}}_1,\ldots,\tilde{\mathcal{L}}_N)}} \mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})].$$

Finally we apply Lemma 6.1 and take the limits $\Lambda \to \infty$, $\epsilon \downarrow 0$, $\hat{\kappa} \downarrow \kappa$, $\tilde{\mathcal{L}}_i \downarrow \mathcal{L}_i$, $i \le N$. The result is

$$V_\kappa(G) \le \sup_{\hat{\mathbb{Q}} \in \mathcal{M}_{\kappa,\mathcal{L}}} \mathbb{E}_{\hat{\mathbb{Q}}}[G(\mathbb{S}^{(1)})].$$

This concludes the proof of the lemma as well as the proof of the main result. □

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